Spectra of positive and negative energies in the linearized NLS problem

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Abstract

We study the spectrum of the linearized NLS equation in three dimensions, in association with the energy spectrum. We prove that unstable eigenvalues of the linearized NLS problem are related to negative eigenvalues of the energy spectrum, while neutrally stable eigenvalues may have both positive and negative energies. The non-singular part of the neutrally stable essential spectrum is always related to the positive energy spectrum. We derive bounds on the number of unstable eigenvalues of the linearized NLS problem and study bifurcations of embedded eigenvalues of positive and negative energies. We develop the L^2-scattering theory for the linearized NLS operators and recover results of Grillakis [G90] with a Fermi Golden Rule.

1 Introduction

In this paper we consider the spectrum of the linearized operator $\mathcal{L} = \mathcal{J}\mathcal{H}$,

$$\mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\Delta + \omega + f(x) & g(x) \\ g(x) & -\Delta + \omega + f(x) \end{pmatrix}, \tag{1.1}$$

where $x \in \mathbb{R}^3$, $\omega > 0$, and $f, g\colon \mathbb{R}^3 \to \mathbb{R}$ are exponentially decaying C^∞-functions. The spectral problem on $L^2(\mathbb{R}^3, \mathbb{C}^2)$,

$$\mathcal{L}\boldsymbol{\psi} = z\boldsymbol{\psi}, \tag{1.2}$$

is related to the linearization of the nonlinear Schrödinger (NLS) equation,

$$i\psi_t = -\Delta\psi + U(x)\psi + F(|\psi|^2)\psi, \tag{1.3}$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and $\psi \in \mathbb{C}$. For suitable functions $U(x)$ and $F(|\psi|^2)$, the NLS equation (1.3) possesses special solutions,

$$\psi = \phi(x)e^{i\omega t}, \tag{1.4}$$

where $\phi: \mathbb{R}^3 \to \mathbb{R}$ and $\phi \in C^\infty$. We assume that $\phi(x)$ is an exponentially decreasing solution of the elliptic problem,

$$-\Delta\phi + \omega\phi + U(x)\phi + F(\phi^2)\phi = 0. \tag{1.5}$$

If $\phi(x) > 0$, $\forall x \in \mathbb{R}^3$, it is referred to as the ground state. A unique radially symmetric ground state exists if $U(x) = 0$ or if $U(x)$ is radially symmetric [M93]. If $\phi(x)$ is sign-indefinite, it is referred to as the excited state.

Linearization of the nonlinear Schrödinger equation (1.3) with the anzats,

$$\psi = \left(\phi(x) + \varphi(x)e^{-izt} + \bar{\theta}(x)e^{i\bar{z}t}\right)e^{i\omega t}, \tag{1.6}$$

where $(\varphi, \theta) : \mathbb{R}^3 \mapsto \mathbb{C}^2$, $z \in \mathbb{C}$, leads to the spectral problem (1.2) with $\boldsymbol{\psi} = (\varphi, \theta)^T$, $f(x) = U(x) + F(\phi^2) + F'(\phi^2)\phi^2$, and $g(x) = F'(\phi^2)\phi^2$. The eigenvalues z of the spectral problem (1.2) are said to be unstable if $\operatorname{Im} z > 0$, neutrally stable if $\operatorname{Im} z = 0$, and stable if $\operatorname{Im} z < 0$. We assume that $U(x) \in C^\infty$ is exponentially decreasing and $F \in C^\infty$, $F(0) = 0$, such that assumptions on $f(x)$, $g(x)$ are satisfied.

The nonlinear Schrödinger equation (1.3) in space of three dimensions was studied recently in context of asymptotic stability of the ground states [C01, TY02a, Per03]. Spectral and orbital stability of the ground states follows from the general theorems of Weinstein [W86] and Grillakis, Shatah, & Strauss [GSS87, GSS90], since $\mathcal{H}$ has a single negative eigenvalue for the positive ground state $\phi(x)$ [S89]. Spectral instabilities of excited states were studied by Jones [J88] and Grillakis [G90] with special instability criteria. Instabilities and radiative decay of the excited states of the NLS equation (1.3) was proved recently by Tsai & Yau [TY03b, T03].

We study spectral properties of the linearized NLS problem (1.2), in context of instabilities of excited states of the NLS equation (1.3). Our main results are based on separation of spectra of positive and negative energies, where the energy functional is defined on $H^1(\mathbb{R}^3, \mathbb{C}^2)$:

$$h = \langle \boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi} \rangle. \tag{1.7}$$

We will be using notations $\langle \mathbf{f}, \mathbf{g} \rangle$ for the vector inner products of $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and notations (f, g) for the scalar inner products of $f, g \in L^2(\mathbb{R}^3, \mathbb{C})$.

Using analysis of constrained eigenvalue problems, we prove that the spectrum of $\mathcal{H}$ with the negative energy (1.7) is related to a subset of isolated or embedded eigenvalues z of the point spectrum of $\mathcal{L}$ corresponding to the eigenvectors $\boldsymbol{\psi}(x)$. This part of the spectrum produces instabilities of the excited states, in which case the linearized NLS problem (1.2) has eigenvalues z with $\operatorname{Im}(z) > 0$. Sharp bounds on the number and type of unstable eigenvalues of the linearized operator $\mathcal{L}$ are given in terms of negative eigenvalues of the energy operator $\mathcal{H}$, see also [Pel03, KKS03]. They improve and generalize the special results obtained in [J88, G90].

Using analysis of wave operators, we prove that the spectrum of $\mathcal{H}$ with the positive energy (1.7) is related to a non-singular part of the essential spectrum of $\mathcal{L}$, as well as to another subset of isolated or embedded eigenvalues z with $\mathrm{Im}(z) = 0$. This part of the spectrum does not produce instabilities of excited states but it leads to instabilities when eigenvalues z with the negative energy (1.7) coalesce with essential spectrum or eigenvalues z with the positive energy (1.7).

Using analysis of a Fermi golden rule, we study the singular part of the essential spectrum, which is related to embedded eigenvalues z with $\mathrm{Im}(z) = 0$ and $|\mathrm{Re}(z)| > \omega$. We prove that embedded eigenvalues z with the positive energy (1.7) disappear under generic perturbation, while the ones with the negative energy (1.7) bifurcate into isolated complex eigenvalues z of the point spectrum of $\mathcal{L}$.

Bifurcations from resonances were recently studied by Kapitula & Sandstede [KS02], who also suggested that instability bifurcations may occur from the interior points of the essential spectrum. We will prove here that these instability bifurcations do not occur in the linearized NLS problem (1.2), since no resonances may occur in the interior points of the essential spectrum of $\mathcal{L}$. The instability bifurcations in the interior points arise therefore only when an embedded eigenvalue with the negative energy (1.7) is supported in the spectrum of $\mathcal{L}$.

Our paper is structured as follows. Main results on spectra of positive and negative energy are formulated in Section 2. Point spectrum of negative energy is studied in Section 3. Non-singular essential spectrum of positive energy is considered in Section 4. Bifurcations of embedded eigenvalues of positive and negative energies are described in Section 5. Section 6 concludes the paper with sharp bounds on the number and type of unstable eigenvalues of $\mathcal{L}$.

2 Main formalism

We use Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

and rewrite $\mathcal{L}$ explicitly as,

$$\mathcal{L} = (-\Delta + \omega + f(x))\,\sigma_3 + ig(x)\sigma_2. \tag{2.2}$$

We notice that $\sigma_1 \mathcal{L} \sigma_1 = -\mathcal{L}$, $\sigma_3 \mathcal{L} \sigma_3 = \mathcal{L}^*$, and $\sigma_j \sigma_k = -\sigma_k \sigma_j$ for $j \neq k$. We also decompose the operator $\mathcal{L}$ into the unbounded differential part $\mathcal{L}_0$ and bounded potential part $V(x)$ as $\mathcal{L} = \mathcal{L}_0 + V(x)$, where $\mathcal{L}_0 = (-\Delta + \omega)\sigma_3$ and $V(x) = f(x)\sigma_3 + ig(x)\sigma_2$.

Assumption 2.1 *Let $V(x) = B^* A$. Then, $A(x)$ and $B(x)$ are continuous, exponentially decaying matrix-valued functions, such that*

$$|A_{i,j}(x)| + |B_{i,j}(x)| < Ce^{-\alpha|x|}, \quad \forall x \in \mathbb{R}^3, \qquad 1 \leq i, j \leq 2, \tag{2.3}$$

for some $\alpha > 0$, $C > 0$.

We denote the point spectrum of $\mathcal{L}$ as $\sigma_p(\mathcal{L})$ and the essential spectrum of $\mathcal{L}$ as $\sigma_e(\mathcal{L})$. The point spectrum is the union of isolated and embedded eigenvalues, while the essential spectrum includes continuous spectrum with resonances and embedded eigenvalues. We use the local L^2_s space defined as

$$L^2_s = \left\{ f : \ (1+|x|^2)^{s/2} f \in L^2 \right\}. \tag{2.4}$$

Before formulating our main results, we shall prove that the operator $\mathcal{L}$ has finitely many eigenvalues and no resonances at interior points of the essential spectrum. For analysis, we use the Birman–Schwinger kernel (see p.89 in [S79]), which was applied to the linearized NLS problem (1.2) by Grillakis [G90, Appendix]. Using a formal substitution $\mathbf{\Psi} = -A\boldsymbol{\psi}$, we can see that the linearized NLS problem,

$$(\mathcal{L}_0 - z)\boldsymbol{\psi} = -B^* A \boldsymbol{\psi}, \tag{2.5}$$

is equivalent to the problem,

$$(\mathcal{I} + \mathcal{Q}_0(z))\,\mathbf{\Psi} = \mathbf{0}, \qquad \mathcal{Q}_0(z) = A\,(\mathcal{L}_0 - z)^{-1}\,B^*, \tag{2.6}$$

where $\mathcal{I}$ is identity matrix in $\mathbb{C}^{2\times 2}$ and $\mathbf{0}$ is the zero vector in $\mathbb{C}^2$.

Proposition 2.2 *The set of isolated and embedded eigenvalues in the spectral problem (2.6) is finite and the dimension of the corresponding generalized eigenspaces is finite.*

Proof. For $\mathrm{Im}\, z \neq 0$, $\mathcal{Q}_0(z)\colon L^2 \to L^2$ is well defined and compact. Denote extension of $\mathcal{Q}_0(z)$ to $\mathcal{D}_+ = \{\mathrm{Im}\, z \geq 0\}$ by $\mathcal{Q}_0^+(z)$. By Agmon [A75], we have:

$$\lim_{|z|\to\infty} \|\mathcal{Q}_0^+(z)\|_{L^2 \mapsto L^2} = 0. \tag{2.7}$$

Then, by Analytic Fredholm Theory, the set of eigenvalues of the operator $(\mathcal{I} + \mathcal{Q}_0^+(z))$ with non-empty generalized kernel $\mathcal{N}_g$ has a zero measure in $\mathcal{D}_+$. This set is finite and $\dim \sum_{z_j} \mathcal{N}_g(\mathcal{I} + \mathcal{Q}_0^+(z_j)) < \infty$, because $A(x)$ and $B(x)$ are exponentially decreasing, see [R78, G90]. In this case, no accumulation points exist near the essential spectrum [R78, G90]. ■

Proposition 2.3 *Let D be the finite set of embedded eigenvalues, E be the set of end points of the essential spectrum, $E = \{\omega\} \cup \{-\omega\}$, and $\sigma_e(\mathcal{L})$ be the essential spectrum, $\sigma_e(\mathcal{L}) = \mathbb{R} - (-\omega, \omega)$, of the spectral problem (2.6). For all $\Omega \in S$, where $S = \sigma_e(\mathcal{L}) - (D \cup E)$, we have $\mathcal{N}_g(\mathcal{I} + \mathcal{Q}_0^+(\Omega + i0)) = 0$.*

The proof of Proposition 2.3 is based on the following lemma.

Lemma 2.4 *Suppose $\Omega \in \mathbb{R}$ and $|\Omega| > \omega$. We have the following maps:*

$$
\begin{aligned}
&(1)\quad \boldsymbol{\psi} \to \boldsymbol{\Psi} = -A\boldsymbol{\psi}, \quad \mathrm{Ker}(\mathcal{L}-\Omega) \subset L^2 \to \mathrm{Ker}(\mathcal{I} + \mathcal{Q}_0^+(\Omega)) \subset L^2 \\
&(2)\quad \boldsymbol{\Psi} \to \boldsymbol{\Phi} = B^*\boldsymbol{\Psi}, \quad \mathrm{Ker}(\mathcal{I} + \mathcal{Q}_0^+(\Omega)) \subset L^2 \to \mathrm{Ker}(\mathcal{I} + V(\mathcal{L}_0 - \Omega - i0)^{-1}) \subset L^2_s,\ s \in \mathbb{R} \\
&(3)\quad \boldsymbol{\Phi} \to \boldsymbol{\psi} = (\mathcal{L}_0 - \Omega - i0)^{-1}\boldsymbol{\Phi}, \quad \mathrm{Ker}(\mathcal{I} + V(\mathcal{L}_0 - \Omega - i0)^{-1}) \subset L^2_s \to \mathrm{Ker}(\mathcal{L}-\Omega) \subset L^2,\ s > \frac{1}{2}
\end{aligned}
$$

Proof. The only nontrivial step is the proof of (3). We follow [RS78, XIII.8] and consider $\Omega > \omega$. The crucial step consists in proving the following two claims:

$$
\boldsymbol{\psi} \in L^2, \qquad \boldsymbol{\psi} \in \mathcal{D}(\mathcal{L}_0) \tag{2.8}
$$

and

$$
(\mathcal{L} - \Omega)\,\boldsymbol{\psi} = \mathbf{0}, \tag{2.9}
$$

such that Ω is an eigenvalue for $\mathcal{L}$. The second claim (2.9) follows from the first claim (2.8), since for $\boldsymbol{\eta} \in C^\infty$:

$$
\langle \boldsymbol{\eta}, \mathcal{L}_0\boldsymbol{\psi}\rangle = \lim_{\epsilon\to 0^+} \langle \boldsymbol{\eta}, \mathcal{L}_0(\mathcal{L}_0 - \Omega - i\epsilon)^{-1}\boldsymbol{\Phi}\rangle = \lim_{\epsilon\to 0^+} \langle \boldsymbol{\eta}, (\Omega + i\epsilon)\boldsymbol{\psi} + \boldsymbol{\Phi}\rangle = \langle \boldsymbol{\eta}, (\Omega - V)\boldsymbol{\psi}\rangle. \tag{2.10}
$$

In order to prove the first claim (2.8), we rewrite explicitly:

$$
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = ((-\Delta + \omega)\sigma_3 - \Omega - i0)^{-1} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \qquad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = -\begin{pmatrix} f & g \\ -g & -f \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.11}
$$

and therefore

$$
(-\Delta + \omega + \Omega)\psi_2 = -g(x)\psi_1 - f(x)\psi_2. \tag{2.12}
$$

Since $\Omega > 0$ and $f(x)$,$g(x)$ are exponentially decreasing, we conclude that $|\psi_2(x)| \le Ce^{-\alpha|x|}$, $\forall x \in \mathbb{R}^3$ for some $\alpha > 0$, $C > 0$ and, therefore, $\psi_2 \in L^2$ and $\psi_2 \in \mathcal{D}(\Delta)$. Furthermore, it follows from Lemma 8 in [RS78, XIII.8] that $\psi_1 \in L^2$ and $\psi_1 \in \mathcal{D}(\Delta)$, if we can prove that

$$
\mathrm{Im}\left((-\Delta + \omega - \Omega - i0)^{-1}\Phi_1, \Phi_1\right) = 0, \tag{2.13}
$$

where (f, g) stands for the scalar inner product of $f, g \in \mathbb{C}$. Using (2.11), we have

$$
\left((-\Delta + \omega - \Omega - i0)^{-1}\Phi_1, \Phi_1\right) = -\left(\psi_1, f\psi_1\right) - \left(\psi_1, g\psi_2\right), \tag{2.14}
$$

and therefore

$$
\mathrm{Im}\left((-\Delta + \omega - \Omega - i0)^{-1}\Phi_1, \Phi_1\right) = -\,\mathrm{Im}\left(\psi_1, g\psi_2\right) = 0, \tag{2.15}
$$

where the last equality follows from (2.12). ∎

Corollary 2.5 *Let $\boldsymbol{\psi}(x)$ be the eigenvector for the embedded eigenvalue $\Omega \in \mathbb{R}$, $\Omega > \omega$ in the problem (2.5). Then, $\boldsymbol{\psi} \in L^2_s$, $s > 0$.*

Proof. It follows from the proof of Lemma 2.4 that $\psi_2 \in L^2_s$, $s > 0$. Since $\Phi_1 \in L^2_s$, $s > 0$, then Theorem IX.41 in [RS78] implies that $\psi_1 \in L^2_s$, $s > 0$. ■

Besides Assumption 2.1, we simplify analysis with more assumptions on the spectrum of the problems (2.5) and (2.6).

Assumption 2.6 *For $\Omega = \pm\omega$, we have $\mathcal{N}_g(\mathcal{I} + \mathcal{Q}_0^+(\Omega + i0)) = 0$.*

Assumption 2.7 $\mathrm{Ker}(\mathcal{L}) = \{\boldsymbol{\varphi}_0\}$ *and $\mathcal{N}_g(\mathcal{L}) = \{\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1\}$, where $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_1$ represent translations of bound states (1.4) along the complex phase $\phi \mapsto e^{i\theta}\phi$, $\theta \in \mathbb{R}$ and the parameter ω, as follows:*

$$\boldsymbol{\varphi}_0 = \begin{pmatrix} \phi(x) \\ -\phi(x) \end{pmatrix}, \qquad \boldsymbol{\varphi}_1 = -\begin{pmatrix} \frac{\partial \phi(x)}{\partial \omega} \\ \frac{\partial \phi(x)}{\partial \omega} \end{pmatrix}. \tag{2.16}$$

Assumption 2.8 *No real eigenvalues z of $\mathcal{L}$ exist, such that $\langle \boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi} \rangle = 0$, where $\boldsymbol{\psi}$ is the eigenvector of $\mathcal{L}$.*

Assumption 2.6 states that the end points $\Omega = \pm\omega$ are neither resonances nor eigenvalues of $\mathcal{L}$. Resonance and eigenvalue at the end points are studies in separate paper [CP04]. Assumption 2.7 states that the kernel of $\mathcal{L}$ is one-dimensional, while the generalized kernel $\mathcal{N}_g(\mathcal{L})$ is two-dimensional, according to the symmetry of the NLS equation (1.3). Although this assumption is somewhat restrictive, we refer to recent paper [CP03] for the case of $\mathcal{N}_g(\mathcal{L})$ of higher algebraic multiplicity. Finally, Assumption 2.8 excludes positive real eigenvalues z with the zero energy (1.7).

Using Assumptions 2.6– 2.8, we consider a decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^2)$ into the $\mathcal{L}$-invariant Jordan blocks:

$$L^2 = \sum_{z \in \sigma_p(\mathcal{L})} \mathcal{N}_g(\mathcal{L} - z) \oplus X_c(\mathcal{L}), \quad X_c(\mathcal{L}) = \left[\sum_{z \in \sigma_p(\mathcal{L})} \mathcal{N}_g(\mathcal{L}^* - z) \right]^{\perp}, \tag{2.17}$$

and, equivalently,

$$L^2 = \sum_{z \in \sigma_p(\mathcal{L})} \mathcal{N}_g(\mathcal{L}^* - z) \oplus X_c(\mathcal{L}^*), \quad X_c(\mathcal{L}^*) = \left[\sum_{z \in \sigma_p(\mathcal{L})} \mathcal{N}_g(\mathcal{L} - z) \right]^{\perp}, \tag{2.18}$$

where $\sigma_p(\mathcal{L}) = \sigma_p(\mathcal{L}^*)$.

Let $n(\mathcal{H})$ be the (finite) number of negative eigenvalues of $\mathcal{H}$ in $L^2(\mathbb{R}^3, \mathbb{C}^2)$, counting with their algebraic multiplicity. Let $n(\mathcal{H})|_X$, $X \subset L^2$ be the number of negative eigenvalues of $\mathcal{PHP}$ in X, where $\mathcal{P} : L^2 \mapsto X$ is an orthogonal projection onto X.

Let N_{real} be the number of positive real eigenvalues of $\mathcal{L}$, N_{imag} be the number of positive imaginary eigenvalues of $\mathcal{L}$, and N_{comp} be the number of complex eigenvalues of $\mathcal{L}$ in the first open quadrant, accounting their multiplicities. It is clear from Assumption 2.7 and standard symmetries of the linearized NLS problem (1.2) that $\dim(\sigma_p(\mathcal{L})) = 2 + 2N_{\rm real} + 2N_{\rm imag} + 4N_{\rm comp}$. It is also understood from Assumptions 2.6 and 2.8 that $N_{\rm real}$ includes both isolated eigenvalues for $0 < \Omega < \omega$ and embedded eigenvalues for $\Omega > \omega$ with the non-zero energy (1.7). Using this setting, we reproduce Theorem 3.1 from [GSS90] and formulate new results on the relations between numbers $n(\mathcal{H})$, $N_{\rm real}$, $N_{\rm imag}$, and $N_{\rm comp}$.

Theorem 1 *Let Assumption 2.7 be satisfied. Then, $Q'(\omega) \neq 0$, where $Q(\omega) = \int_{\mathbb{R}^3} \phi^2(x)dx$ is the squared L^2-norm of the standing wave solution (1.4). Let $X_0(\mathcal{L})$ be the constrained subspace of $L^2(\mathbb{R}^3, \mathbb{C}^2)$:*

$$X_0(\mathcal{L}) = \{\boldsymbol{\psi} \in L^2 : \langle \boldsymbol{\psi}, \boldsymbol{\varphi}_0^* \rangle = 0, \ \langle \boldsymbol{\psi}, \boldsymbol{\varphi}_1^* \rangle = 0\}, \tag{2.19}$$

where $\boldsymbol{\varphi}_j^* = \sigma_3 \boldsymbol{\varphi}_j$, $j = 0, 1$. Then $n(\mathcal{H})|_{X_0} = n(\mathcal{H}) - 1$ if $Q'(\omega) > 0$ and $n(\mathcal{H})|_{X_0} = n(\mathcal{H})$ if $Q'(\omega) < 0$.

Theorem 2 *Let Assumption 2.8 be satisfied. Let $N_{\rm real}^-$ and $N_{\rm real}^+$ be the number of positive real eigenvalues of $\mathcal{L}$ corresponding to eigenvectors $\boldsymbol{\psi}(x)$ with the negative and positive energy (1.7), respectively, such that $N_{\rm real} = N_{\rm real}^- + N_{\rm real}^+$. Let $X_c(\mathcal{L})$ be the non-singular part of the essential spectrum of $\mathcal{L}$ in (2.17). Then*

$$n(\mathcal{H})|_{X_c} = n(\mathcal{H})|_{X_0} - 2N_{\rm real}^- - N_{\rm imag} - 2N_{\rm comp}. \tag{2.20}$$

Theorem 3 *Let Assumptions 2.1, 2.6, and 2.8 be satisfied. The energy functional (1.7) is strictly positive quadratic form in $X_c(\mathcal{L})$:*

$$\langle \boldsymbol{\psi}, \mathcal{H} \boldsymbol{\psi} \rangle > 0, \qquad \boldsymbol{\psi} \in X_c(\mathcal{L}). \tag{2.21}$$

Theorem 1 and 2 are proved in Section 3, while Theorem 3 is proved in Section 4. These two results lead to the closure relation between the negative index of $\mathcal{H}$ and the eigenvalues of $\mathcal{L}$ in the linearized NLS problem (1.2).

Corollary 2.9 *Let Assumptions 2.1, 2.6, 2.7, and 2.8 be satisfied. Then, the following closure relation is true:*

$$N_{\rm imag} + 2N_{\rm comp} + 2N_{\rm real}^- = n(\mathcal{H}) - p(Q'), \tag{2.22}$$

where $p(Q') = 1$ if $Q'(\omega) > 0$ and $p(Q') = 0$ if $Q'(\omega) < 0$.

Corollary 2.9 can be used in tracing bifurcations of unstable eigenvalues $N_{\rm comp}$ and $N_{\rm imag}$ in the linearized NLS problem (1.2) by parameter continuations [KKS03]. The closure relation (2.22) was first formulated in [Pel03] for the matrix linearized NLS equation on $x \in \mathbb{R}$. It was proved in [Pel03] with the Sylvester's Inertia Law of matrix analysis. Our analysis here does not use matrix analysis but relies on functional analysis of energy operators and constrained quadratic forms.

3 Point spectrum of negative energy

We focus here on the point spectrum $\sigma_p(\mathcal{L})$, which consists of a finite set of isolated and embedded eigenvalues of finite multiplicities. We show that a subset of the point spectrum of $\mathcal{L}$ in the linearized NLS problem (1.2) is related to the spectrum of $\mathcal{H}$ with the negative energy (1.7). For simplicity, we work with simple eigenvalues and discuss the general case of multiple eigenvalues in the end of this section.

For our analysis, we conveniently rewrite the eigenvalue problem (1.2) in the equivalent form,

$$\begin{cases} \mathcal{L}_+ u = zw \\ \mathcal{L}_- w = zu \ , \end{cases} \tag{3.1}$$

where z is the eigenvalue and $\mathbf{u} = (u, w)^T$ is the eigenvector. The new problem (3.1) follows from the linearized NLS problem (1.2) with $\boldsymbol{\psi} = (u + w, u - w)^T$ and $\mathcal{L}_\pm = -\Delta + \omega + f(x) \pm g(x)$. The energy functional (1.7) is equivalently written as

$$h = 2(u, \mathcal{L}_+ u) + 2(w, \mathcal{L}_- w), \tag{3.2}$$

where (f, g) stands for the scalar inner product of $f, g \in \mathbb{C}$. We consider separately the cases of real, purely imaginary and complex eigenvalues z.

Let $z = z_0 \neq 0$ be a simple real eigenvalue of the problem (3.1) with the eigenvector $(u_0, w_0)^T$. It is obvious that the problem (3.1) has another simple eigenvalue $z = -z_0$ with the eigenvector $(u_0, -w_0)^T$. The adjoint problem also has eigenvalues z_0 and $-z_0$, with the eigenvectors $(w_0, u_0)^T$ and $(-w_0, u_0)^T$, respectively. It follows from the system (3.1) that

$$(u_0, \mathcal{L}_+ u_0) = z_0 (w_0, u_0) = (w_0, \mathcal{L}_- w_0). \tag{3.3}$$

We have the following lemma.

Lemma 3.1 *Let $z = z_0 \neq 0$ be a simple real eigenvalue of $\mathcal{L}$. The quadratic forms $(u_0, \mathcal{L}_+ u_0) = (w_0, \mathcal{L}_- w_0)$ are non-zero.*

Proof. Suppose $(u_0, \mathcal{L}_+ u_0) = (w_0, \mathcal{L}_- w_0) = 0$. By the Fredholm Alternative, there exists an eigenvector $(u_0^{(1)}, w_0^{(1)})^T \in \mathcal{N}_g(\mathcal{L} - z_0)$, which satisfies the nonhomogeneous equation,

$$\mathcal{L}_+ u_0^{(1)} = z_0 w_0^{(1)} + w_0, \qquad \mathcal{L}_- w_0^{(1)} = z_0 u_0^{(1)} + u_0. \tag{3.4}$$

However, dimension of $\mathcal{N}_g(\mathcal{L} - z_0)$ must be one, by the assumption that $z = z_0$ is a simple eigenvalue. The contradiction is resolved when $(u_0, \mathcal{L}_+ u_0) = (w_0, \mathcal{L}_- w_0) \neq 0$. ∎

Let $z = iz_I \neq 0$ be a simple purely imaginary eigenvalue of the problem (3.1) with the eigenvector $(u_R + iu_I, w_R + iw_I)^T$. We show that we can set $u_I = 0$, $w_R = 0$. Indeed, the spectral problem (3.1) is rewritten with $z = iz_I$ as

$$\left\{ \begin{array}{rcl} \mathcal{L}_+ u_R &=& -z_I w_I \\ \mathcal{L}_- w_I &=& z_I u_R \end{array} \right. \qquad \left\{ \begin{array}{rcl} \mathcal{L}_+ u_I &=& z_I w_R \\ \mathcal{L}_- w_R &=& -z_I u_I \end{array} \right. \tag{3.5}$$

Since $z = iz_I$ is a simple eigenvalue, the vectors $(u_R, w_I)^T$ and $(u_I, -w_R)^T$ are linearly dependent, so that we can set $u_I = 0$, $w_R = 0$. Thus, the problem (3.1) has the eigenvalue $z = iz_I$ with the eigenvector $(u_R, iw_I)^T$. It also has the eigenvalue $z = -iz_I$ with the eigenvector $(u_R, -iw_I)^T$. The adjoint problem has the eigenvalues iz_I and $-iz_I$ with the eigenvectors $(iw_I, u_R)^T$ and $(-iw_I, u_R)^T$ respectively. It follows from (3.5) that

$$(u_R, \mathcal{L}_+ u_R) = -z_I (w_I, u_R) = -(w_I, \mathcal{L}_- w_I). \tag{3.6}$$

The proof of Lemma 3.1 implies that $(u_R, \mathcal{L}_+ u_R) = -(w_I, \mathcal{L}_- w_I) \neq 0$, if $z = iz_I \neq 0$ is a simple eigenvalue of $\mathcal{L}$.

Let $z = z_R + iz_I$, such that $z_R, z_I \neq 0$ be a simple complex eigenvalue of the problem (3.1) with the eigenvector $(u_R + iu_I, w_R + iw_I)^T$. Components of the eigenvector are coupled by the system of equations:

$$\left\{ \begin{array}{rcl} \mathcal{L}_+ u_R &=& z_R w_R - z_I w_I \\ \mathcal{L}_+ u_I &=& z_I w_R + z_R w_I \end{array} \right. \tag{3.7}$$

and

$$\left\{ \begin{array}{rcl} \mathcal{L}_- w_R &=& z_R u_R - z_I u_I \\ \mathcal{L}_- w_I &=& z_I u_R + z_R u_I \end{array} \right. \tag{3.8}$$

It is obvious that the problem (3.1) has three other eigenvalues $\bar{z}$, $-z$ and $-\bar{z}$ with the eigenvectors $(u_R - iu_I, w_R - iw_I)^T$, $(u_R + iu_I, -w_R - iw_I)^T$ and $(u_R - iu_I, -w_R + iw_I)^T$, respectively. The adjoint problem has the same four eigenvalues with the adjoint eigenvectors $(w, u)^T$.

Using the decomposition

$$\left(\begin{array}{c} u \\ w \end{array} \right) = c_1 \left(\begin{array}{c} u_R \\ w_R \end{array} \right) + c_2 \left(\begin{array}{c} u_I \\ w_I \end{array} \right),$$

where (c_1, c_2) are arbitrary parameters, we show that quadratic forms $(u, \mathcal{L}_+ u)$ and $(w, \mathcal{L}_- w)$ have one positive and one negative eigenvalues. The quadratic forms transform as follows:

$$(u, \mathcal{L}_+ u) = \langle \mathbf{c}, M_+ \mathbf{c} \rangle, \qquad (w, \mathcal{L}_- w) = \langle \mathbf{c}, M_- \mathbf{c} \rangle,$$

where $\mathbf{c} = (c_1, c_2)^T \in \mathbb{R}^2$ and the matrices $M_\pm$ take the form:

$$M_+ = \left(\begin{array}{cc} (u_R, \mathcal{L}_+ u_R) & (u_R, \mathcal{L}_+ u_I) \\ (u_I, \mathcal{L}_+ u_R) & (u_I, \mathcal{L}_+ u_I) \end{array} \right) \tag{3.9}$$

and

$$M_- = \left(\begin{array}{cc} (w_R, \mathcal{L}_- w_R) & (w_R, \mathcal{L}_- w_I) \\ (w_I, \mathcal{L}_- w_R) & (w_I, \mathcal{L}_- w_I) \end{array} \right). \tag{3.10}$$

Lemma 3.2 *Let $z = z_R + iz_I$ be a simple complex eigenvalue, such that $z_R, z_I \neq 0$. Matrices $M_\pm$ have one positive and one negative eigenvalues. Moreover, $M_+ = M_-$.*

Proof. We derive two relations from (3.7) and (3.8):

$$z_R(w_R, u_I) - z_I(w_I, u_I) = z_R(w_I, u_R) + z_I(w_R, u_R), \tag{3.11}$$

$$z_R(w_I, u_R) - z_I(w_I, u_I) = z_R(w_R, u_I) + z_I(w_R, u_R). \tag{3.12}$$

Since $z_R, z_I \neq 0$, we have relations:

$$(w_I, u_I) = -(w_R, u_R), \quad (w_I, u_R) = (w_R, u_I). \tag{3.13}$$

It follows from (3.13) that

$$(u_I, \mathcal{L}_+ u_I) = -(u_R, \mathcal{L}_+ u_R), \quad (u_I, \mathcal{L}_+ u_R) = (u_R, \mathcal{L}_+ u_I), \tag{3.14}$$

and therefore, $\mathrm{tr}(M_+) = 0$ and $\det(M_+) = -(u_R, \mathcal{L}_+ u_R)^2 - (u_R, \mathcal{L}_+ u_I)^2 \leq 0$. By Fredholm Alternative, we have $\det(M_+) \neq 0$, if $z = z_R + iz_I$ is a simple complex eigenvalue. Therefore, $\det(M_+) < 0$ and matrices $M_+ = M_-$ have one positive and one negative eigenvalues. ∎

Let $K = N_{\mathrm{real}} + N_{\mathrm{imag}} + 2N_{\mathrm{comp}}$ and assume that all non-zero eigenvalues of $\mathcal{L}$ are simple. We define the linear constrained subspace $X_c(\mathcal{L})$ from orthogonality conditions:

$$X_c = \{\boldsymbol{\psi} \in X_0(\mathcal{L}) : \ \{\langle \boldsymbol{\psi}, \boldsymbol{\varphi}_j^* \rangle = 0\}_{j=1}^{2K}\}, \tag{3.15}$$

where $\boldsymbol{\varphi}_j^*$ are adjoint vectors to the eigenvector $\boldsymbol{\varphi}_j$ of the problem (1.2) with $z = z_j$. Equivalently, we rewrite the spaces $X_0(\mathcal{L})$ and $X_c(\mathcal{L})$ for the vector $\mathbf{u} = (u, w)^T \in \mathbb{C}^2$:

$$\hat{X}_0 = \left\{ \mathbf{u} \in L^2 : \ (u, \phi) = 0, \ (w, \phi_\omega) = 0 \right\}, \tag{3.16}$$

$$\hat{X}_c = \left\{ \mathbf{u} \in \hat{X}_0(\mathcal{L}) : \ \{(u, w_j) = 0, \ (w, u_j) = 0\}_{j=1}^{K} \right\}, \tag{3.17}$$

where $\phi_\omega = \partial_\omega \phi(x)$. The index $1 \leq j \leq K$ runs through denote all eigenvalues of $\sigma_p(\mathcal{L})$. Abusing notations, we understand that $\mathbf{u}_j = (u_0, w_0)^T$ is the eigenvector for a simple real eigenvalue $z = z_0$, $\mathbf{u}_j = (u_R, iw_I)^T$ is the eigenvector for a simple purely imaginary eigenvalue $z = iz_I$, and $\mathbf{u}_j = (u_R, w_R)^T$, $\mathbf{u}_{j+1} = (u_I, w_I)^T$ are eigenvectors for a simple complex eigenvalue $z = z_R + iz_I$. Using the same notations, we prove that the eigenvectors $\mathbf{u}_j$ for distinct eigenvalues $z = z_j$ are orthogonal with respect to the adjoint eigenvectors $\mathbf{u}_j^* = \sigma_1 \mathbf{u}_j$.

Lemma 3.3 *Let z_i and z_j be two eigenvalues of the problem (3.1) with two eigenvectors $(u_i, w_i)^T$ and $(u_j, w_j)^T$, such that $z_i \neq \pm z_j$ and $z_i \neq \pm \bar{z}_j$. Components of the eigenvectors are linearly independent and skew-orthogonal as follows:*

$$(u_i, w_j) = (w_i, u_j) = 0. \tag{3.18}$$

Proof. Orthogonality relations (3.18) follow from the system (3.1) if $\mathbf{u}_i, \mathbf{u}_j \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and $z_i \neq \pm z_j$, $z_i \neq \pm \bar{z}_j$. Linear independence of $(u_i, w_i)^T$ and $(u_j, w_j)^T$ is standard for distinct eigenvalues. Furthermore, each separate set (u_i, u_j) and (w_i, w_j) is linearly independent, if $z_i \neq \pm z_j$, $z_i \neq \pm \bar{z}_j$, and $z_i, z_j \neq 0$. ∎

We prove Theorems 1 and 2, based on the following abstract lemma.

Lemma 3.4 *Let L be a self-adjoint operator on a Hilbert space $X \subset L^2$ with a finite negative index $n(L)|_X$, empty kernel, and positive essential spectrum. Let X_c be the constrained linear subspace,*

$$X_c = \left\{ v \in X : \ \{(v, v_j) = 0\}_{j=1}^N \right\}, \tag{3.19}$$

where the set $\{v_j\}_{j=1}^N$, $v_j \in X$ is linearly independent. Let negative eigenvalues of L in X_c be defined by the problem,

$$Lv = \mu v - \sum_{j=1}^N \nu_j v_j, \quad v \in X_c, \quad \mu < 0, \tag{3.20}$$

where $\{\nu_j\}_{j=1}^N$ is a set of Lagrange multipliers. Let the matrix-valued function $A(\mu)$ be defined in the form:

$$A_{i,j}(\mu) = (v_i, (\mu - L)^{-1} v_j), \quad \mu \notin \sigma(L). \tag{3.21}$$

If P eigenvalues of $A(0)$ are non-negative, $0 \leq P \leq N$, then $n(L)|_{X_c} = n(L)|_X - P$.

Before proving Lemma 3.4, we consider the following elementary fact.

Lemma 3.5 *If B is a negative definite operator on a Hilbert space X, $(v, Bv) < 0$ for all $v \in X$, $v \neq 0$, and the set $\{v_j\}_{j=1}^N \in X$ is linearly independent, then the matrix $\mathcal{B}$, such that*

$$\mathcal{B}_{i,j} = (v_i, B v_j), \qquad 1 \leq i, j \leq N, \tag{3.22}$$

is a negative definite matrix on $\mathbb{C}^N$.

Proof. For any $\mathbf{x} = (x_1, ..., x_N)^T \in \mathbb{C}^N$, $\mathbf{x} \neq \mathbf{0}$, the matrix $\mathcal{B}$ is negative definite, since

$$\langle \mathbf{x}, \mathcal{B}\mathbf{x} \rangle = \left(\sum_{j=1}^N x_j v_j, B \sum_{j=1}^N x_j v_j \right) < 0, \tag{3.23}$$

and the set $\{v_j\}_{j=1}^N$ is linearly independent, such that $\sum_{j=1}^N x_j v_j \neq 0$. ∎

Proof of Lemma 3.4. Via spectral calculus (see [S71, RS72]) we have the decomposition in $X \subset L^2$:

$$A_{i,j}(\mu) = \int_{\mu_1}^{\infty} \frac{(v_i, dE_\lambda v_j)}{\mu - \lambda}, \quad \mu \notin \sigma(L), \tag{3.24}$$

where E_λ is the spectral family associated with the operator L, μ_1 is the smallest eigenvalue in X, and $\sigma(L)$ is the spectrum of L in X. An easy calculation yields

$$\begin{aligned}\Big|\frac{A_{i,j}(\mu+h)-A_{i,j}(\mu)}{h}+\int_{\mu_1}^{\infty}\frac{(v_i,dE_\lambda v_j)}{(\mu-\lambda)^2}\Big| &\leq |h|\|v_i\|_X\Big\|\int_{\mu_1}^{\infty}\frac{dE_\lambda}{(\mu-\lambda)^2(\mu+h-\lambda)}\Big\|_X\|v_j\|_X \\ &\leq \frac{|h|}{d(\mu)^2(d(\mu)-|h|)}\|v_i\|_X\|v_j\|_X, \end{aligned} \tag{3.25}$$

where $d(\mu)=\min\{|\mu-\lambda|,\ \lambda\in\sigma(L)\}$. Since the upper bound in (3.25) vanishes in the limit $h\to 0$, the derivative $A'_{i,j}(\mu)$ exists,

$$A'_{i,j}(\mu)=-\int_{\mu_1}^{\infty}\frac{(v_i,dE_\lambda v_j)}{(\mu-\lambda)^2}=-(v_i,(\mu-L)^{-2}v_j). \tag{3.26}$$

The operator $-(\mu-L)^{-2}$ is negative definite, such that the matrix $A'(\mu)$ is negative definite on $\mathbb{C}^N$ by Lemma 3.5. Let $\{\alpha_i(\mu)\}_{i=1}^N$ be real-valued eigenvalues of $A(\mu)$ and $\{\mathbf{v}_i(\mu)\}_{i=1}^N$ be eigenvectors of $A(\mu)$. According to the perturbation theory, the derivatives $\alpha'_i(\mu)$ are given by eigenvalues of the matrix $\langle\mathbf{v}_i(\mu),A'(\mu)\mathbf{v}_j(\mu)\rangle$. Since $A'(\mu)$ is negative definite, we have $\alpha'_i(\mu)<0$, $1\leq i\leq N$ by Lemma 3.5. Therefore $\{\alpha_i(\mu)\}_{i=1}^N$ are monotonically decreasing functions for $\mu\notin\sigma(L)$.

Let $Lv_k^j=\mu_k v_k^j$, $\mu_k<0$, $1\leq k\leq K_0$, $1\leq j\leq m_k$, where m_k is the multiplicity of μ_k and $\{v_k^j\}_{j=1}^{m_k}$ is the orthonormal set of eigenfunctions for μ_k. The negative index of the operator L in X is $n(L)|_X=\sum_{k=1}^{K_0}m_k\equiv K$. Via spectral calculus, we have

$$A(\mu)=\frac{1}{\mu-\mu_k}A_k+B_k(\mu)\ , \tag{3.27}$$

where

$$(A_k)_{i,j} = (P_k v_i,P_k v_j),\qquad 1\leq i,j\leq N, \tag{3.28}$$

$$(B_k)_{i,j}(\mu) = \int_{[\mu_1,\infty)\backslash(\mu_k-\delta,\mu_k+\delta)}\frac{(v_i,dE_\lambda v_j)}{\mu-\lambda},\qquad 1\leq i,j\leq N, \tag{3.29}$$

and P_k is the projection onto the subspace spanned by $\{v_k^j\}_{j=1}^{m_k}$. It is clear that there exists δ, such that no other eigenvalues of operator L occur in the interval $(\mu_k-\delta,\mu_k+\delta)$. The n-th derivative for the (i,j)-element of the matrix $B_k(\mu)$ is

$$\frac{d^n}{d\mu^n}(B_k)_{i,j}(\mu)=(-1)^n n!\int_{[\mu_1,\infty)\backslash(\mu_k-\delta,\mu_k+\delta)}\frac{(v_i,dE_\lambda v_j)}{(\mu-\lambda)^{n+1}},\quad 1\leq i,j\leq N,$$

where n is any nonnegative integer, and $\mu\in(\mu_k-\frac{1}{2}\delta,\mu_k+\frac{1}{2}\delta)$. Therefore,

$$\begin{aligned}\Big|\frac{d^n}{d\mu^n}(B_k)_{i,j}(\mu)\Big| &\leq n!\ \ \Big|\Big(v_i,\int_{[\mu_1,\infty)\backslash(\mu_k-\delta,\mu_k+\delta)}\frac{dE_\lambda}{(\mu-\lambda)^{n+1}}v_j\Big)\Big| \\ &\leq n!\|v_i\|_X\Big\|\int_{[\mu_1,\infty)\backslash(\mu_k-\delta,\mu_k+\delta)}\frac{dE_\lambda}{(\mu-\lambda)^{n+1}}\Big\|_X\|v_j\|_X, \end{aligned} \tag{3.30}$$

by Schwarz inequality and definition of operator norm. Furthermore, we have the estimate for $\mu \in (\mu_k - \frac{1}{2}\delta, \mu_k + \frac{1}{2}\delta)$:

$$\left\| \int_{[\mu_1,\infty)\setminus(\mu_k-\delta,\mu_k+\delta)} \frac{dE_\lambda}{(\mu-\lambda)^{n+1}} \right\|_X = \sup\left\{ \frac{1}{|\mu-\lambda|^{n+1}},\ \lambda \in \sigma(L),\ \lambda \neq \mu_k \right\} \leq \left(\frac{2}{\delta}\right)^{n+1}, \tag{3.31}$$

and therefore

$$\left| \frac{d^n}{d\mu^n}(B_k)_{i,j}(\mu) \right| \leq n! \left(\frac{2}{\delta}\right)^{n+1} \|v_i\|_X \|v_j\|_X, \quad \mu \in \left(\mu_k - \frac{\delta}{2}, \mu_k + \frac{\delta}{2}\right). \tag{3.32}$$

We conclude that the infinitely-differentiable matrix-valued function $B_k(\mu)$ is analytic in the neighborhood of $\mu = \mu_k$, by comparison with geometric series. The representation (3.27) implies that the eigenvalue problem for $A(\mu)$ can be written as

$$(A_k + B_k(\mu)(\mu - \mu_k))\,\mathbf{v}_i(\mu) = \alpha_i(\mu)(\mu - \mu_k)\mathbf{v}_i(\mu), \quad 1 \leq i \leq N.$$

The Hermitian matrix $A_k + B_k(\mu)(\mu - \mu_k)$ is analytic in the neighborhood of $\mu = \mu_k$ and therefore, according to perturbation theory (see, e.g., [RS78]), its eigenvalues are analytic in the neighborhood of $\mu = \mu_k$, such that

$$\alpha_i(\mu)(\mu - \mu_k) = \alpha_i^0 + (\mu - \mu_k)\beta_i(\mu), \qquad 1 \leq i \leq N,$$

where α_i^0 are eigenvalues of A_k and $\beta_i(\mu)$ are analytic near $\mu = \mu_k$. Therefore,

$$\alpha_i(\mu) = \frac{\alpha_i^0}{\mu - \mu_k} + \beta_i(\mu), \quad 1 \leq i \leq N. \tag{3.33}$$

near $\mu = \mu_k$, which means that the behavior of $\alpha_i(\mu)$ in the neighborhood of $\mu = \mu_k$ depends on the rank of the matrix A_k. Since the matrix A_k is non-negative for all $\mathbf{x} \in \mathbb{C}^N$,

$$\langle \mathbf{x}, A_k \mathbf{x} \rangle = \left\| P_k \sum_{i=1}^{N} x_i v_i \right\|_X^2 \geq 0,$$

then $\alpha_i^0 \geq 0$ $1 \leq i \leq N$. Given $N_k = \mathrm{rank}(A_k)$, such that $0 \leq N_k \leq \min(m_k, N)$, there are precisely N_k linearly independent $\{P_k v_i\}_{i=1}^{N_k}$. Then, we can construct the orthonormal set of eigenvectors $\{v_k^i\}_{i=1}^{m_k}$ corresponding to μ_k, such that $v_k^i \notin X_c$, $1 \leq i \leq N_k$ and $v_k^i \in X_c$, $N_k + 1 \leq i \leq m_k$. Therefore, $\alpha_i^0 > 0$, $1 \leq i \leq N_k$, such that

$$\lim_{\mu \to \mu_k^+} \alpha_i(\mu) = +\infty, \qquad \lim_{\mu \to \mu_k^-} \alpha_i(\mu) = -\infty, \qquad 1 \leq i \leq N_k,$$

while $\alpha_i^0 = 0$, $N_k + 1 \leq i \leq N$, such that $\alpha_i(\mu)$, $N_k + 1 \leq i \leq N$ are continuous at $\mu = \mu_k$.

Assume now that $\alpha_i(\mu)$, $1 \leq i \leq N$ has vertical asymptotes at $\mu = \mu_{i_1} < \mu_{i_2} < \ldots < \mu_{i_{K_i}} < 0$. Each element of the matrix $A(\mu)$ in (3.21) can be estimated using Schwarz inequality and spectral calculus for $\mu < \mu_1$:

$$|(v_i, (\mu - L)^{-1} v_j)| \leq \|v_i\|_X \left(\int_{\mu_1}^{\infty} \frac{(v_j, dE_\lambda v_j)}{(\mu - \lambda)^2} \right)^{1/2} \leq \frac{\|v_i\|_X \|v_j\|_X}{|\mu| - |\mu_1|},$$

such that the eigenvalues $\alpha_i(\mu)$, $1 \leq i \leq N$ tend to zero as $\mu \to -\infty$. Since they are monotonically decreasing when $\mu \notin \sigma(L)$ we have $\alpha_i(\mu) < 0$ on $\mu \in (-\infty, \mu_{i_1})$.

On the interval $\mu \in (\mu_{i_l}, \mu_{i_{l+1}})$, $1 \leq l \leq K_i - 1$, the eigenvalue $\alpha_i(\mu)$ is continuous, monotonic and has a simple zero at $\mu = \mu_l^*$, $\mu_l^* \in (\mu_{i_l}, \mu_{i_{l+1}})$ with the unique eigenfunction,

$$v_l^* = \sum_{j=1}^{N} \nu_j^l (\mu_l^* - L)^{-1} v_j,$$

where $A(\mu_l^*)\mathbf{v}_i(\mu_l^*) = \mathbf{0}$ and $\mathbf{v}_i(\mu_l^*) = (\nu_1^l, ..., \nu_N^l)^T$. Since

$$\|(\mu_l^* - L)^{-1} v_i\|_X^2 = \int_{\mu_1}^{\infty} \frac{(v_i, dE_\lambda v_i)}{(\mu_l^* - \lambda)^2} \leq \frac{\|v_i\|_X^2}{d(\mu_l^*, \sigma(L))^2} < \infty, \quad 1 \leq i \leq N,$$

where $d(\mu_l^*) = \min\{|\mu_l^* - \lambda|, \ \lambda \in \sigma(L)\}$, we prove that $\|u_l^*\|_X < \infty$. Therefore, $(K_i - 1)$ negative eigenvalues of L in X_c are located at $\mu_l^* \in (\mu_{i_l}, \mu_{i_{l+1}})$, $1 \leq l \leq K_i - 1$. Due to the monotonicity, $\alpha_i(\mu) > 0$ on $\mu \in (\mu_{i_{K_i}}, 0)$ if $\alpha_i(0) \geq 0$ or has precisely one zero at $\mu = \mu_{K_i}^*$, $\mu_{K_i}^* \in (\mu_{i_{K_i}}, 0)$ if $\alpha_j(0) < 0$.

The negative index of the operator L in the constrained subspace X_c is $n(L)|_{X_c} = K - \sum_{i=1}^N (K_i - K_i + 1 - \Theta(-\alpha_i(0))) = n(L)|_X - P$, where $\Theta(x)$ is the Heaviside step-function and P is the number of non-negative eigenvalues $\alpha_i(0)$, $1 \leq i \leq N$, such that $0 \leq P \leq N$. ∎

Proof of Theorem 1. By Assumption 2.7, there is no generalized eigenvector $\boldsymbol{\varphi}_2$ to the eigenvector $\boldsymbol{\varphi}_1$ in (2.16). By Fredholm Alternative Theorem, it implies that

$$(\phi, \phi_\omega) = \frac{1}{2} Q'(\omega) \neq 0. \tag{3.34}$$

Consider the self-adjoint operator $\mathrm{diag}(\mathcal{L}_+, \mathcal{L}_-)$ on $L^2(\mathbb{R}^3, \mathbb{C}^2)$. Define negative eigenvalues of $\mathcal{L}_\pm$ in $\hat{X}_0(\mathcal{L})$ by the constrained problem:

$$\mathcal{L}_+ u = \mu u - \nu_0^+ \phi, \quad \mathcal{L}_- w = \mu w - \nu_0^- \phi_\omega, \quad (u, w)^T \in \hat{X}_0, \quad \mu < 0, \tag{3.35}$$

where $\nu_0^\pm$ are Lagrange multipliers. Then, we apply Lemma 3.4 with a single constraint ($N = 1$) and compute

$$A^+(0) = -(\phi, \mathcal{L}_+^{-1} \phi) = \frac{1}{2} Q'(\omega). \tag{3.36}$$

Using (3.34) and $\mathcal{L}_- \phi = 0$, we prove that $A^-(0)$ is unbounded. Since $A^-(\mu)$ is monotonically decreasing for $\mu \notin \sigma(\mathcal{L}_-)$, we have

$$\lim_{\mu \to 0^-} A^-(\mu) = -\infty. \tag{3.37}$$

Extending the last paragraph of the proof of Lemma 3.4 to the case when $\lim_{\mu \to 0^-} \alpha_i(\mu) = -\infty$, we prove the statement of Theorem. ∎

Proof of Theorem 2. Consider the self-adjoint operator $\mathrm{diag}(\mathcal{L}_+, \mathcal{L}_-)$ on a subspace $\hat{X}_0(\mathcal{L}) \subset L^2(\mathbb{R}^3, \mathbb{C}^2)$, which is defined by (3.16). Define negative eigenvalues of $\mathcal{L}_\pm$ in $\hat{X}_c(\mathcal{L})$ by the constrained problem:

$$\mathcal{L}_+ u = \mu u - \sum_{j=1}^{K} \nu_j^+ P_0^+ w_j, \quad \mathcal{L}_- w = \mu w - \sum_{j=1}^{K} \nu_j^- P_0^- u_j, \quad (u, w)^T \in \hat{X}_c, \quad \mu < 0, \tag{3.38}$$

where $\{\nu_j^\pm\}_{j=1}^K$ is a set of Lagrange multipliers and $P_0^\pm$ is orthogonal projection from $L^2(\mathbb{R}^3, \mathbb{C}^2)$ to $\hat{X}_0(\mathcal{L}_\pm)$. By Lemma 3.3, components of the eigenvectors $(u_j, w_j)^T$, $1 \le j \le K$ are linearly independent and skew-orthogonal to components of eigenvectors $(0, \phi)$ and $(\phi_\omega, 0)$ for the zero eigenvalue, such that $(w_j, \phi_\omega) = 0$, $P_0^+ w_j = w_j$ and $(u_j, \phi) = 0$, $P_0^- u_j = u_j$, $1 \le j \le K$.

Define matrices $A^\pm(\mu)$ by the elements:

$$A_{i,j}^+(\mu) = \left(w_i, (\mu - \mathcal{L}_+)^{-1} w_j\right), \qquad A_{i,j}^-(\mu) = \left(u_i, (\mu - \mathcal{L}_-)^{-1} u_j\right). \tag{3.39}$$

It follows from the orthogonality relations (3.18) that the matrices $A^\pm(0)$ are decomposed into diagonal blocks. For the real eigenvalue $z = z_0$, the blocks include the diagonal entry:

$$A_{j,j}^+(0) = A_{j,j}^-(0) = -\frac{1}{z_0}(u_0, w_0) = -\frac{1}{z_0^2}(u_0, \mathcal{L}_+ u_0) = -\frac{1}{z_0^2}(w_0, \mathcal{L}_- w_0). \tag{3.40}$$

For the purely imaginary eigenvalues $z = i z_I$, the blocks include the diagonal entry:

$$A_{j,j}^+(0) = -A_{j,j}^-(0) = \frac{1}{z_I}(u_R, w_I) = -\frac{1}{z_I^2}(u_R, \mathcal{L}_+ u_R) = \frac{1}{z_I^2}(w_I, \mathcal{L}_- w_I). \tag{3.41}$$

For the complex eigenvalue $z = z_R + i z_I$, the blocks include the 2-by-2 matrix $M_+ = M_-$ defined in (3.9)–(3.10):

$$A_{i,k}^+(0) = A_{i,k}^-(0) = -(Z^2 M_+)_{I,J}, \tag{3.42}$$

where $j \le i, k \le j+1$, $1 \le I, J \le 2$, and

$$Z = \frac{1}{z_R^2 + z_I^2} \begin{pmatrix} z_R & z_I \\ -z_I & z_R \end{pmatrix}.$$

Since $\det(Z^2 M_+) = \det(M_+)/(z_R^2 + z_I^2)^2 < 0$, the matrices $Z^2 M_+$ has one negative and one negative eigenvalue, similarly to matrix M_+. Counting together, the matrices $A^\pm(0)$ have $2N_{\mathrm{real}}^- + N_{\mathrm{imag}} + 2N_{\mathrm{comp}}$ positive eigenvalues. Therefore, the reduction formula (2.20) is proved by Lemma 3.4. ∎

Remark 3.6 *The proof of Theorem 2 is given in case of simple eigenvalues. Generalization for semi-simple eigenvalues is trivial. Multiple eigenvalues can be considered as the limiting case of simple eigenvalues. Multiple purely imaginary and complex eigenvalues preserve the relation (2.20) in the limiting case, while multiple real eigenvalues with zero energy (3.2) violate the relation (2.20) in the limiting case. We excluded the latter eigenvalues by Assumption 2.8 to simplify the formalism.*

4 Non-singular essential spectrum of positive energy

We focus here on the action of $\mathcal{L}$ in $X_c(\mathcal{L})$, where $X_c(\mathcal{L})$ is the non-singular part of the essential spectrum of $\mathcal{L}$, defined equivalently in (2.17) and (3.15). We show that the non-singular essential spectrum of $\mathcal{L}$ is related to the spectrum of $\mathcal{H}$ with the positive energy (1.7).

We prove Theorem 3 using scattering theory of wave operators in $X_c(\mathcal{L})$. From technical point, we apply the theory of global smoothness by Kato [K66] and prove that the operator $\mathcal{L}$ acts in $X_c(\mathcal{L})$ like the operator $\mathcal{L}_0$ acts in $L^2(\mathbb{R}^3, \mathbb{C}^2)$. The concept of global smoothness for the proof of existence and completeness of wave operators cannot be used in many classical situations, e.g. for short-range Schrödinger operators on the line. In some situations, a local smoothness can be used instead, see [RS78, Theorem XIII.7C]. The local smoothness applies to the operator $\mathcal{L}$, which does not meet Assumption 2.6, as shown in the separate paper [CP04]. We formulate the main result on existence of wave operators in $X_c(\mathcal{L})$.

Proposition 4.1 *Let Assumptions 2.1, 2.6, and 2.8 be satisfied. Then, there exist isomorphisms between Hilbert spaces $W\colon L^2 \mapsto X_c(\mathcal{L})$ and $Z\colon X_c(\mathcal{L}) \mapsto L^2$, which are inverse of each other, defined as follows:*

$$\forall\, u \in L^2, \forall\, v \in X_c(\mathcal{L}^*) :$$
$$\langle Wu, v\rangle = \langle u, v\rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle A(\mathcal{L}_0 - \lambda - i\epsilon)^{-1} u, B(\mathcal{L}^* - \lambda + i\epsilon)^{-1} v\rangle d\lambda, \tag{4.1}$$

and

$$\forall\, u \in X_c(\mathcal{L}), \forall\, v \in L^2 :$$
$$\langle Zu, v\rangle = \langle u, v\rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle A(\mathcal{L} - \lambda - i\epsilon)^{-1} u, B(\mathcal{L}_0 - \lambda + i\epsilon)^{-1} v\rangle d\lambda. \tag{4.2}$$

By Kato [K66], Proposition 4.1 is proved with two lemmas below.

Lemma 4.2 *There exists $c > 0$ such that $\forall\, u \in L^2$ and $\forall\, \epsilon \neq 0$, the following bounds are true:*

$$\int_{-\infty}^{\infty} \|A(\mathcal{L}_0 - i\epsilon - \lambda)^{-1} u\|^2 d\lambda \leq c\|u\|^2 \tag{4.3}$$
$$\int_{-\infty}^{\infty} \|B(\mathcal{L}_0 - i\epsilon - \lambda)^{-1} u\|^2 d\lambda \leq c\|u\|^2. \tag{4.4}$$

Proof. See Corollary to Theorem XIII.25 in [RS78] for the proof. ∎

Lemma 4.3 *There exists $c > 0$ such that $\forall\, \epsilon \neq 0$, the following bounds are true:*

$$\int_{-\infty}^{\infty} \|B(\mathcal{L}^* - i\epsilon - \lambda)^{-1} u\|^2 d\lambda \leq c\|u\|^2, \quad \forall\, u \in X_c(\mathcal{L}^*) \tag{4.5}$$
$$\int_{-\infty}^{\infty} \|A(\mathcal{L} - i\epsilon - \lambda)^{-1} u\|^2 d\lambda \leq c\|u\|^2, \quad \forall\, u \in X_c(\mathcal{L}). \tag{4.6}$$

Proof. We prove the second bound (4.6). The proof of the first bound (4.5) can be done similarly. We write

$$A(\mathcal{L}-z)^{-1}\mathbf{v} = (\mathcal{I}+\mathcal{Q}_0^+(z))^{-1}A(\mathcal{L}_0-z)^{-1}\mathbf{v}. \tag{4.7}$$

If $(\mathcal{I}+\mathcal{Q}_0^+(z))^{-1}$ is uniformly bounded in $z \in \mathcal{D}$, there is nothing to prove. By Propositions 2.2, 2.3 and by Assumption 2.6, this operator is unbounded only near isolated and embedded eigenvalues of $\mathcal{L}$. If z_0 is an isolated eigenvalue of $\mathcal{L}$, then $A(\mathcal{L}-z)^{-1}\mathbf{v}$ is analytic near $z = z_0$ if $\mathbf{v} \in X_c(\mathcal{L})$ because the orthogonal projection of $\mathbf{v}$ in $\mathcal{N}_g(\mathcal{L}-z_0)$ is empty. We show that similar arguments can be developed for embedded eigenvalues. By Assumption 2.8, the embedded eigenvalue $z = \Omega_0$ has a non-zero energy (1.7).

Suppose $z = \Omega_0 > \omega$ is an embedded eigenvalue of $\mathcal{L}$. For simplicity we assume that $\mathrm{dimKer}(\mathcal{L}-\Omega_0) = 1$, such there exist $\boldsymbol{\phi}_0$ and $\boldsymbol{\phi}_0^*$:

$$(\mathcal{L}-\Omega_0)\boldsymbol{\phi}_0 = \mathbf{0}, \quad (\mathcal{L}^*-\Omega_0)\boldsymbol{\phi}_0^* = \mathbf{0}, \quad \langle \boldsymbol{\phi}_0, \boldsymbol{\phi}_0^* \rangle = 1. \tag{4.8}$$

It follows from the explicit form (1.1) that

$$\boldsymbol{\phi}_0^* = \frac{\sigma_3 \boldsymbol{\phi}_0}{\langle \boldsymbol{\phi}_0, \sigma_3 \boldsymbol{\phi}_0 \rangle}. \tag{4.9}$$

By Assumption 2.8, $\langle \boldsymbol{\phi}_0, \sigma_3 \boldsymbol{\phi}_0 \rangle \neq 0$, which is equivalent to the condition that $\mathrm{dim}\mathcal{N}_g(\mathcal{L}-\Omega_0) = 1$. The embedded eigenvalue $z = \Omega_0$ is a singular point for $(\mathcal{I}+\mathcal{Q}_0^+(z))^{-1}$ with the Laurent expansion,

$$(\mathcal{I}+\mathcal{Q}_0^+(z))^{-1} = \sum_{l=0}^{M-1} (z-\Omega_0)^{-M+l}\mathcal{C}_{-M+l} + \mathcal{F}(z), \tag{4.10}$$

where $\mathcal{F}(z)$ is analytic around $z = \Omega_0$ and $\mathcal{C}_{-M+l}$ are finite rank operators for some $M < \infty$, see [S68]. We have

$$(\mathcal{I}+\mathcal{Q}_0^+(\Omega_0))\mathcal{C}_{-M} = 0, \quad (\mathcal{I}+\mathcal{Q}_0^+(\Omega_0))^*\mathcal{C}_{-M}^* = 0, \tag{4.11}$$

and Lemma 2.4 implies that

$$\mathcal{C}_{-M} = c_0 A\boldsymbol{\phi}_0 \langle \cdot\,, B\boldsymbol{\phi}_0^* \rangle, \tag{4.12}$$

where c_0 is a constant. We show that $M = 1$ and $c_0 = 1$. The operator $A(\mathcal{L}-z)^{-1}\boldsymbol{\phi}_0$ has the main term $(\Omega_0 - z)^{-1}A\boldsymbol{\phi}_0$ in the Laurent expansion at $z = \Omega_0$, while the operator $(\mathcal{I}+\mathcal{Q}_0^+(z))^{-1}A(\mathcal{L}_0 - z)^{-1}\boldsymbol{\phi}_0$ has the main term $-c_0(z-\Omega_0)^{-M}A\boldsymbol{\phi}_0$. While $A\boldsymbol{\phi}_0 \neq 0$ by Lemma 2.4, it follows from (4.7) that the two terms must be the same, such that $M = 1$ and $c_0 = 1$.

A uniform expansion of $A(\mathcal{L}-z)^{-1}\mathbf{v}$ in z near $z = \Omega_0$ follows from (4.7) and (4.10):

$$A(\mathcal{L}-z)^{-1}\mathbf{v} = (z-\Omega_0)^{-1}A\boldsymbol{\phi}_0\langle A(\mathcal{L}_0-z)^{-1}\mathbf{v}, B\boldsymbol{\phi}_0^*\rangle + \mathcal{F}(z)A(\mathcal{L}_0-z)^{-1}\mathbf{v}. \tag{4.13}$$

Since operator $\mathcal{F}(z)$ is bounded in z, the second term of (4.13) is in Hardy space $H^2(\mathcal{D}_+)$, where $\mathcal{D}_+ = \{z \in \mathbb{C} : \ \mathrm{Im}\, z \geq 0\}$. We analyze the singular part, given by the first term of (4.13):

$$\begin{aligned} & (z-\Omega_0)^{-1}A\boldsymbol{\phi}_0\langle A\left[\mathcal{R}_0(z) - \mathcal{R}_0(\Omega_0)\right]\mathbf{v}, B\boldsymbol{\phi}_0^*\rangle + (z-\Omega_0)^{-1}A\boldsymbol{\phi}_0\langle A\mathcal{R}_0(\Omega_0)\mathbf{v}, B\boldsymbol{\phi}_0^*\rangle \\ = \ & A\boldsymbol{\phi}_0\langle A\mathcal{R}_0(\Omega_0)\mathcal{R}_0(z)\mathbf{v}, B\boldsymbol{\phi}_0^*\rangle - (z-\Omega_0)^{-1}A\boldsymbol{\phi}_0\langle \mathbf{v}, \boldsymbol{\phi}_0^*\rangle, \end{aligned} \tag{4.14}$$

where $\mathcal{R}_0(z) = (\mathcal{L}_0 - z)^{-1}$ and we have used that

$$\mathcal{R}_0(z) - \mathcal{R}_0(\Omega_0) = (z - \Omega_0)\mathcal{R}_0(\Omega_0)\mathcal{R}_0(z).$$

If $\mathbf{v} \in X_c(\mathcal{L})$, then $\langle \mathbf{v}, \boldsymbol{\phi}_0^* \rangle = 0$ and

$$A(\mathcal{L} - z)^{-1}\mathbf{v} = -A\boldsymbol{\phi}_0 \langle \mathcal{R}_0(z)\mathbf{v}, \boldsymbol{\phi}_0^* \rangle + \mathcal{F}(z)A(\mathcal{L}_0 - z)^{-1}\mathbf{v}. \tag{4.15}$$

Due to Corollary 2.5, $\boldsymbol{\phi}_0(x)$ and $\boldsymbol{\phi}_0^*(x)$ are rapidly decreasing, such that $\langle \mathcal{R}_0(z)\mathbf{v}, \boldsymbol{\phi}_0^* \rangle$ is in Hardy space $H^2(\mathcal{D}_+)$, and so is $A(\mathcal{L} - z)^{-1}\mathbf{v}$. ∎

Proof of Theorem 3. Let P_c and P_c^* be the spectral projection on $X_c(\mathcal{L})$ and $X_c(\mathcal{L}^*)$, respectively. Then,

$$P_c^*\sigma_3 = \sigma_3 P_c, \quad W^*\sigma_3 = \sigma_3 Z, \quad Z^*\sigma_3 = \sigma_3 W, \quad Z\mathcal{L} = \mathcal{L}_0 Z. \tag{4.16}$$

If $\boldsymbol{\psi} \in X_c(\mathcal{L})$, there exists $\hat{\boldsymbol{\psi}} \in L^2$, such that $\boldsymbol{\psi} = W\hat{\boldsymbol{\psi}}$. Therefore, a simple transformation shows that

$$\langle \boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi} \rangle = \langle W\hat{\boldsymbol{\psi}}, \sigma_3\mathcal{L}W\hat{\boldsymbol{\psi}} \rangle = \langle W\hat{\boldsymbol{\psi}}, \mathcal{L}^* Z^*\sigma_3\hat{\boldsymbol{\psi}} \rangle = \langle \mathcal{L}_0 ZW\hat{\boldsymbol{\psi}}, \sigma_3\hat{\boldsymbol{\psi}} \rangle = \langle (-\Delta + \omega)\mathcal{I}\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\psi}} \rangle > 0. \tag{4.17}$$

∎

Corollary 4.4 *There exists a constant C such that for all $t > 0$:*

$$\| e^{i\mathcal{L}t} : \; X_c(\mathcal{L}) \to X_c(\mathcal{L}) \| < C. \tag{4.18}$$

Corollary 4.4 is taken as a hypothesis in the recent paper [RSS03], while the arguments leading to the statement in the original paper [C01] are inconclusive. The statement is proved trivially, in the context of Proposition 4.1.

5 Embedded eigenvalues of positive and negative energies

We focus here on the singular part of the essential spectrum of $\mathcal{L}$. Since embedded resonances are impossible due to Proposition 2.3, the singular part is only related to the embedded eigenvalues of the point spectrum $z = \Omega_0$, where $|\Omega_0| > \omega$. The embedded eigenvalues are structurally unstable, so that a generic perturbation with a non-zero Fermi Golden rule results in bifurcations of embedded eigenvalues off the essential spectrum. We show that the embedded eigenvalues of $\mathcal{L}$ with the positive energy (1.7) disappear from the point spectrum of $\mathcal{L}$, while the embedded eigenvalues of $\mathcal{L}$ with the negative energy (1.7) become isolated complex eigenvalues of the point spectrum of $\mathcal{L}$. These results are in agreement with our main results, formulated in Theorems 2 and 3, since the non-singular essential spectrum of $\mathcal{L}$ has the positive energy (1.7), while complex eigenvalues of $\mathcal{L}$ are related to the spectrum of $\mathcal{H}$ with the negative energy (1.7).

Embedded eigenvalues with the negative energy are very typical in the linearized NLS problem (1.2), since the diagonal part of operator $\mathcal{L}$ takes the form of the pair of Schrödinger operators $\mathcal{L}_s$ and $(-\mathcal{L}_s)$, where $\mathcal{L}_s = -\Delta + \omega + f(x)$, pointing in the opposite directions. When $\phi = 0$, we have $f = U(x)$ and $g = 0$, such that negative eigenvalues of $\mathcal{L}_s$ become embedded eigenvalues with the negative energy in the linearized NLS problem (1.2), see also [TY02a, TY03b].

Instability of embedded eigenvalues with the negative energy was shown for the linearized NLS problem (1.2) with variational arguments by Grillakis [G90, Theorem 2.4]. Recently, Tsai and Yau [TY03b] proved the same results with the Fermi Golden rule arguments. Soffer and Weinstein [SW98] also used the time-dependent resonance theory with the Fermi Golden rule. The concept of the Fermi Golden rule is related to rigorous methods used in literature of late 60's, see [H70, H74].

Following to Howland [H70, H74], we frame the problems treated in [G90, TY03b] in a general context and show that the analysis involving Weinstein–Aronszajin determinant in [H70, H74] contains all essential elements in the proof of structural instability of embedded eigenvalues. We strengthen Theorem 2.4 of [G90], by allowing the embedded eigenvalues to have the positive energy (1.7) as well. Also our Assumptions 2.1, 2.6 and 2.8 are weaker than Assumption $(*)$ in [G90, p. 320]. Assumptions of [TY03b] satisfy both our Assumptions and Assumption $(*)$. Our main result is formulated in the following proposition.

Proposition 5.1 *Let Assumptions 2.1, 2.6 and 2.8 be satisfied. Assume that wave operators* $W : L^2 \mapsto X_c(\mathcal{L})$ *and* $Z : X_c(\mathcal{L}) \mapsto L^2$ *for the unperturbed operator* $\mathcal{L} = \mathcal{L}_0 + V(x)$ *exist. Let* $\mathcal{L}_1 = \mathcal{L} + \epsilon V_1(x)$, *where* $V_1(x) = B_1^* A_1$, *and* A_1, B_1 *are smooth functions that satisfy the decay rate (2.3). Let* $z = \Omega_0 > \omega$ *be a semi-simple embedded eigenvalue of* $\mathcal{L}$, *such that* $\mathrm{dimKer}(\mathcal{L} - \Omega_0) = \dim(\mathcal{N}_g(\mathcal{L} - \Omega_0)) = N$ *with the basis of eigenvectors* $\{\boldsymbol{\phi}_j\}_{j=1}^N$. *Suppose that* $\langle \boldsymbol{\phi}_j, \mathcal{H}\boldsymbol{\phi}_j \rangle < 0$ *for* $1 \leq j \leq k$ *and* $\langle \boldsymbol{\phi}_j, \mathcal{H}\boldsymbol{\phi}_j \rangle > 0$ *for* $k+1 \leq j \leq N$. *Then, for generic* $V_1(x)$ *and for small* ϵ, *the point spectrum of* $\mathcal{L}_1$ *has exactly* $2k$ *complex conjugate eigenvalues* z *with* $\mathrm{Im}(z) \neq 0$ *and no embedded eigenvalues near* $z = \Omega_0$.

Remark 5.2 *We assume in Proposition 5.1 that the potential functions* $A_1(x)$ *and* $B_1(x)$ *satisfy the same condition (2.3) as the potential functions* $A(x)$ *and* $B(x)$. *The assumption on existence and completeness of wave operators* $W : L^2 \mapsto X_c(\mathcal{L})$ *and* $Z : X_c(\mathcal{L}) \mapsto L^2$ *is proved for the unperturbed operator* $\mathcal{L} = \mathcal{L}_0 + \epsilon V(x)$ *in Proposition 4.1.*

The proof of Proposition 5.1 is based on two results of Howland [H70, H74]. First, we relate locations of embedded eigenvalues with zeros of an analytic function $\Delta(z)$, which is the Weinstein–Aronszajin determinant [H70]. Then, we look for zeros of $\Delta(z)$ at small ϵ and use the Fermi Golden rule [H74].

Lemma 5.3 *Let $\mathcal{Q}^+(z) = A_1(\mathcal{L}-z)^{-1}B_1^*$ and $\mathcal{Q}_1^+(z) = A_1(\mathcal{L}_1-z)^{-1}B_1^*$ be operator extensions in $\mathcal{D}_+ = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. Let $z_0 \in \mathcal{D}_+$, $z_0 \neq \pm\omega$ be an eigenvalue of $\mathcal{L}$. Then $\mathcal{Q}^+(z)$ and $\mathcal{Q}_1^+(z)$ are meromorphic in a neighborhood of $z = z_0$ with the principal part $A_1(\mathcal{L}-z)^{-1}|_{\mathcal{N}_g(\mathcal{L}-z_0)}B_1^*$ and $A_1(\mathcal{L}_1-z)^{-1}|_{\mathcal{N}_g(\mathcal{L}_1-z_0)}B_1^*$, respectively.*

Proof. Let $A_1 = A$. It follows from (4.7) for $\mathcal{L} = \mathcal{L}_0 + B^*A$ that

$$A(\mathcal{L}-z)^{-1}B_1^* = (\mathcal{I} + B^*(\mathcal{L}_0-z)^{-1}A)^{-1}A(\mathcal{L}_0-z)^{-1}B_1^*, \tag{5.1}$$

where $A(\mathcal{L}_0-z)^{-1}B_1^*$ is analytic around $z = z_0$ and $(\mathcal{I}+B^*(\mathcal{L}_0-z)^{-1}A)^{-1}$ is meromorphic around z_0. As a result, $A(\mathcal{L}-z)^{-1}B_1^*$ is meromorphic around $z = z_0$. Denote $\mathcal{P}$ as the projection onto $N_g(\mathcal{L}-z_0)$, according to the decomposition (2.17):

$$A(\mathcal{L}-z)^{-1}B_1^* = A(\mathcal{L}-z)^{-1}|_{N_g(\mathcal{L}-z_0)}B_1^* + A(\mathcal{I}-\mathcal{P})(\mathcal{L}-z)^{-1}B_1^*. \tag{5.2}$$

Since $A(\mathcal{L}-z)^{-1}B_1^*\mathbf{v}$ and $A\mathcal{P}(\mathcal{L}-z)^{-1}B_1^*\mathbf{v}$ are meromorphic near any $z_0 \in \mathbb{R}$, $z_0 \neq \pm\omega$, then $A(\mathcal{I}-\mathcal{P})(\mathcal{L}-z)^{-1}B_1^*\mathbf{v}$ is meromorphic. Let $\tilde{\mathbf{v}} = (\mathcal{I}-\mathcal{P})B_1^*\mathbf{v}$, such that $\tilde{\mathbf{v}} \in X_c(\mathcal{L})$. By Lemma 4.3, $A(\mathcal{L}-z)^{-1}\tilde{\mathbf{v}}$ is in Hardy space $H^2(\mathcal{D}_+)$. Therefore, $z_0 \in \mathbb{R}$, $z_0 \neq \pm\omega$ can not be a pole for $A(\mathcal{I}-\mathcal{P})(\mathcal{L}-z)^{-1}B_1^*\mathbf{v}$, so it is analytic around $z = z_0$.

Let $A_1 \neq A$ and choose the factorizations $V = B^*A$ and $V_1 = B_1^*A_1$ so that $A_1A^{-1} \in L^\infty$. Then, Lemma is proved as

$$A_1(\mathcal{L}-z)^{-1}B_1^* = A_1A^{-1}A(\mathcal{L}-z)^{-1}B_1^*. \tag{5.3}$$

Similarly, let $\mathcal{L}_1 = \mathcal{L}_0 + V_2$, where $V_2 = B_2^*A_2$ and A_2, B_2 are smooth functions that satisfy the decay rate (2.3), such that $A_1A_2^{-1} \in L^\infty(\mathbb{R}^3)$. Then, we write

$$A_1(\mathcal{L}_1-z)^{-1}B_1^* = A_1A_2^{-1}A_2(\mathcal{L}_1-z)^{-1}B_1^*, \tag{5.4}$$

where $A_2(\mathcal{L}_1-z)^{-1}B_1^*$ satisfies the analogue of (5.1). As a result, the last statement of Lemma follows from the same arguments, which are used for $\mathcal{L}$ and applied now to $\mathcal{L}_1$. ■

By Theorem 1 in [S68] and also Lemma 1.4 in [H70], Lemma 5.3 implies that there exists an analytic operator valued function $\mathcal{A}(z)$ such that

$$\mathcal{A}(z)(\mathcal{I} + \mathcal{Q}^+(z)) = \mathcal{I} + \mathcal{F}(z), \tag{5.5}$$

where $\mathcal{F}(z)$ is meromorphic of finite rank. The Weinstein–Aronszajin determinant $\Delta(z)$ is defined in [K76, p.161] as

$$\Delta(z) = \det(\mathcal{I} + \mathcal{F}(z)). \tag{5.6}$$

The function $\Delta(z)$ is meromorphic and complex-valued in $z \in \mathcal{D}_+$.

Lemma 5.4 *Let $\nu(z,\mathcal{L}) = \dim(\mathcal{N}_g(\mathcal{L}-z))$ and $\nu(z,\mathcal{L}_1) = \dim(\mathcal{N}_g(\mathcal{L}_1-z))$ in $z \in \mathcal{D}_+$. Let $\nu(z,\Delta)$ be the index of $\Delta(z)$, such that $\nu(z,\Delta) = k$ if z is a zero of order k, $\nu(z,\Delta) = -k$ if z is a pole of order k, and $\nu(z,\Delta) = 0$ otherwise. If $z = \Omega_0 > \omega$ is an embedded eigenvalue of $\mathcal{L}$, then*

$$\nu(\Omega_0,\mathcal{L}_1) = \nu(\Omega_0,\mathcal{L}) + \nu(\Omega_0,\Delta). \tag{5.7}$$

Proof. Denote $\mathcal{P}$ and $\mathcal{P}_1$ as the projections on $\mathcal{N}_g(\mathcal{L}-\Omega_0)$ and $\mathcal{N}_g(\mathcal{L}_1-\Omega_0)$, respectively, associated to the Jordan block decomposition. Let $\mathcal{D}_0$ be a small disk centered at $z = \Omega_0$. By calculations in [H70], we have

$$\begin{aligned} \nu(\Omega_0,\Delta) &= \frac{1}{2\pi i}\mathrm{Tr}\int_{\partial\mathcal{D}_0} \frac{d}{dz}\mathcal{Q}^+(z)\left(\mathcal{I}+\mathcal{Q}^+(z)\right)^{-1} dz \\ &= \frac{1}{2\pi i}\mathrm{Tr}\int_{\partial\mathcal{D}_0} A_1(\mathcal{L}-z)^{-1}(\mathcal{L}_1-z)^{-1}B_1^* dz, \end{aligned} \tag{5.8}$$

where Tr stands for trace, defined in [K76, p.162]. We prove that the representation (5.8) is equivalent to

$$\begin{aligned} \nu(\Omega_0,\Delta) &= \mathrm{TrRes}\left[A_1\mathcal{P}(\mathcal{L}-z)^{-1}(\mathcal{L}_1-z)^{-1}B_1^*,\Omega_0\right] \\ &+ \mathrm{TrRes}\left[A_1(\mathcal{L}-z)^{-1}(\mathcal{L}_1-z)^{-1}\mathcal{P}_1B_1^*,\Omega_0\right], \end{aligned} \tag{5.9}$$

where Res stands for residue. It is clear that

$$\begin{aligned} &\mathrm{Res}\left[A_1(\mathcal{L}-z)^{-2}B_1^*(\mathcal{I}+A_1(\mathcal{L}-z)^{-1}B_1^*)^{-1},\Omega_0\right] \\ = &-\mathrm{Res}\left[A_1(\mathcal{L}-z)^{-2}B_1^*A_1(\mathcal{L}-z)^{-1}B_1^*(\mathcal{I}+A_1(\mathcal{L}-z)^{-1}B_1^*)^{-1},\Omega_0\right] \\ = &-\mathrm{Res}\left[A_1(\mathcal{L}-z)^{-2}B_1^*A_1(\mathcal{L}_1-z)^{-1}B_1^*,\Omega_0\right]. \end{aligned} \tag{5.10}$$

Given two analytic functions $F(z)$ and $G(z)$ in a Banach algebra, with principal parts F_{sing} and G_{sing} at a given point $z = z_0$, then,

$$\mathrm{Res}[FG,z_0] = \mathrm{Res}[F_{\mathrm{sing}}G,z_0] + \mathrm{Res}[FG_{\mathrm{sing}},z_0].$$

Using this formula and Lemma 5.3, we transform (5.10) to the form:

$$-\mathrm{Res}\left[A_1\mathcal{P}(\mathcal{L}-z)^{-2}B_1^*A_1(\mathcal{L}_1-z)^{-1}B_1^*,\Omega_0\right] - \mathrm{Res}\left[A_1(\mathcal{L}-z)^{-2}B_1^*A_1(\mathcal{L}_1-z)^{-1}\mathcal{P}_1B_1^*,\Omega_0\right],$$

which is the right-hand-side of (5.9). Using the formula,

$$(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1} = (\mathcal{L}_1-z)^{-1}B_1^*A_1(\mathcal{L}-z)^{-1},$$

we have

$$\mathcal{P}\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P} = \mathcal{P}(\mathcal{L}-z)^{-1}B_1^*(\mathcal{I}+\mathcal{Q}^+(z))^{-1}A_1(\mathcal{L}-z)^{-1}\mathcal{P} \tag{5.11}$$

$$\mathcal{P}_1\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P}_1 = \mathcal{P}_1(\mathcal{L}_1-z)^{-1}B_1^*(\mathcal{I}+\mathcal{Q}_1^+(z))^{-1}A_1(\mathcal{L}_1-z)^{-1}\mathcal{P}_1. \tag{5.12}$$

It follows from Lemma 5.3 that the right-hand-sides of (5.11) and (5.12) are meromorphic around $z = \Omega_0$. Next, let $\mathcal{A}(z)$ and $\mathcal{B}(z)$ be operator valued functions, which are meromorphic at $z = 0$, such that

$$\mathcal{A}(z) = \sum_{k\in\mathbb{Z}} \mathcal{A}_k z^k, \qquad \mathcal{B}(z) = \sum_{k\in\mathbb{Z}} \mathcal{B}_k z^k,$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are of finite rank for all $k \in \mathbb{Z}$. Then, we have

$$\mathrm{Tr}\,\mathrm{Res}\left[\mathcal{A}(z)\mathcal{B}(z),0\right] = \mathrm{Tr}\left[\sum_{k\in\mathbb{Z}} \mathcal{A}_k\mathcal{B}_{-k-1}\right] = \mathrm{Tr}\left[\sum_{k\in\mathbb{Z}} \mathcal{B}_k\mathcal{A}_{-k-1}\right] = \mathrm{Tr}\,\mathrm{Res}\left[\mathcal{B}(z)\mathcal{A}(z),0\right].$$

As a result,

$$\begin{aligned}\nu(\Omega_0,\Delta) &= \mathrm{Tr\ Res}\left[\mathcal{P}\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P},\Omega_0\right] \\ &+ \mathrm{Tr\ Res}\left[\mathcal{P}_1\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P}_1,\Omega_0\right]. \end{aligned} \tag{5.13}$$

Since the right-hand-sides of (5.11) and (5.12) are meromorphic at $z = \Omega_0$ and

$$\begin{aligned}\mathcal{P}(\mathcal{L}-z)^{-1} &= (\mathcal{L}-z)^{-1}\mathcal{P} = (\mathcal{L}-z)^{-1}|_{\mathcal{N}_g(\mathcal{L}-\Omega_0)}, \\ \mathcal{P}_1(\mathcal{L}_1-z)^{-1} &= (\mathcal{L}_1-z)^{-1}\mathcal{P}_1 = (\mathcal{L}_1-z)^{-1}|_{\mathcal{N}_g(\mathcal{L}_1-\Omega_0)},\end{aligned}$$

we conclude that $\mathcal{P}(\mathcal{L}_1-z)^{-1}\mathcal{P}$ and $\mathcal{P}_1(\mathcal{L}-z)^{-1}\mathcal{P}_1$ are meromorphic functions around $z = \Omega_0$. By an elementary approximation argument, Lemma 5.3 implies that

$$\mathcal{P}(\mathcal{P}-\mathcal{P}_1)\mathcal{P} = -\mathrm{Res}\left[\mathcal{P}\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P},\Omega_0\right] \tag{5.14}$$

$$\mathcal{P}_1(\mathcal{P}-\mathcal{P}_1)\mathcal{P}_1 = -\mathrm{Res}\left[\mathcal{P}_1\left[(\mathcal{L}-z)^{-1} - (\mathcal{L}_1-z)^{-1}\right]\mathcal{P}_1,\Omega_0\right], \tag{5.15}$$

such that

$$\nu(\Omega_0,\Delta) = -\mathrm{Tr}\left[\mathcal{P}(\mathcal{P}-\mathcal{P}_1)\mathcal{P}\right] - \mathrm{Tr}\left[\mathcal{P}_1(\mathcal{P}-\mathcal{P}_1)\mathcal{P}_1\right] = \mathrm{Tr}\mathcal{P}_1 - \mathrm{Tr}\mathcal{P}, \tag{5.16}$$

which is equivalent to (5.7). ■

The concluding lemma applies the result of Lemma 5.4 to perturbation expansions near the embedded eigenvalue $z = \Omega_0$.

Lemma 5.5 *For generic $V_1(x)$, the degeneracy of the embedded eigenvalue $z = \Omega_0$ breaks and the perturbed eigenvalues $z_j(\epsilon)$, $1 \le j \le N$ are analytic at $\epsilon = 0$ and coincide with eigenvalues of the matrix:*

$$\hat{\Delta}_{i,j}(\epsilon) = \Omega_0\delta_{i,j} + \epsilon\langle A_1\boldsymbol{\phi}_i, B_1\boldsymbol{\phi}_j^*\rangle - \epsilon^2\langle \mathcal{Q}_c^+(\Omega_0)A_1\boldsymbol{\phi}_i, B_1\boldsymbol{\phi}_j^*\rangle + \mathrm{O}(\epsilon^3), \tag{5.17}$$

where

$$\mathcal{Q}_c^+(z) = \mathcal{Q}^+(z) - (\Omega_0 - z)^{-1}\sum_{j=1}^N A_1\boldsymbol{\phi}_j\langle\,\cdot\,,\boldsymbol{\phi}_j^*\rangle B_1^*, \tag{5.18}$$

and $\{\boldsymbol{\phi}_j^*\}_{j=1}^N$ is the basis in $\mathrm{Ker}(\mathcal{L}^* - \Omega_0)$, such that $\langle\boldsymbol{\phi}_i,\boldsymbol{\phi}_j^*\rangle = \delta_{i,j}$.

Proof. It follows from (5.18) that

$$(\mathcal{I} + \epsilon \mathcal{Q}_c^+(z))^{-1}(\mathcal{I} + \epsilon \mathcal{Q}^+(z)) = \mathcal{I} + \mathcal{F}(z), \tag{5.19}$$

where the meromorphic finite-rank operator $\mathcal{F}(z)$ is given explicitly by:

$$\mathcal{F}(z) = (\mathcal{I} + \epsilon \mathcal{Q}_c^+(z))^{-1} \epsilon (\Omega_0 - z)^{-1} \sum_{j=1}^{N} A_1 \boldsymbol{\phi}_j \langle \cdot, \boldsymbol{\phi}_j^* \rangle B_1^*. \tag{5.20}$$

Denote $\hat{\Delta}(z, \epsilon) = (\Omega_0 - z)^N \Delta(z, \epsilon) = (\Omega_0 - z)^N \det(\mathcal{I} + \mathcal{F}(z))$. Using the identity

$$\det(\mathcal{I} + \mathcal{A}\mathcal{B}) = \det(\mathcal{I} + \mathcal{B}\mathcal{A})$$

for any finite-rank operators $\mathcal{A}$ and $\mathcal{B}$, we reduce $\hat{\Delta}(z, \epsilon)$ to the form:

$$\hat{\Delta}(z, \epsilon) = \det \left(\Omega_0 - z + \epsilon \sum_{j=1}^{N} A_1 \boldsymbol{\phi}_j \langle (\mathcal{I} + \epsilon \mathcal{Q}_c^+(z))^{-1} \cdot, B_1 \boldsymbol{\phi}_j^* \rangle \right). \tag{5.21}$$

The determinant $\hat{\Delta}(z, \epsilon)$ is calculated by restricting to the space $\{A_1 \boldsymbol{\phi}_j\}_{j=1}^N$ and with the perturbation series expansions, resulting in (5.17). ∎

Proof of Proposition 5.1. The proof follows to that of Theorem 2.1 in [H74]. We use the relation between the sets $\{\boldsymbol{\phi}_j\}_{j=1}^N$ and $\{\boldsymbol{\phi}_j^*\}_{j=1}^N$:

$$\boldsymbol{\phi}_j^* = \frac{\sigma_3 \boldsymbol{\phi}_j}{\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle}, \tag{5.22}$$

where $\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle \neq 0$ for a semi-simple eigenvalue $z = \Omega_0$. We consider the imaginary part of the matrix $\hat{\Delta}_{i,j}(\epsilon)$ in (5.17):

$$\operatorname{Im} \hat{\Delta}_{i,j}(\epsilon) = -\epsilon^2 \frac{\operatorname{Im} \langle \mathcal{Q}_c^+(\Omega_0) A_1 \boldsymbol{\phi}_i, B_1 \sigma_3 \boldsymbol{\phi}_j \rangle}{\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle} + \mathrm{O}(\epsilon^3), \tag{5.23}$$

where $\mathcal{Q}_c^+(z)$ follows from (5.18) as

$$\mathcal{Q}_c^+(z) = A_1 \left(\sum_{z_j \in \sigma_p \backslash \Omega_0} P_{z_j} + P_c \right) (\mathcal{L} - z)^{-1} B_1^*. \tag{5.24}$$

The projections P_{z_j} to the point spectrum $\sigma_p(\mathcal{L})$ do not affect the imaginary part of (5.23), since the operator

$$(\Omega_0 - z_j)^{-1} \sigma_3 P_{z_j} + (\Omega_0 - \bar{z}_j)^{-1} \sigma_3 P_{\bar{z}_j}$$

is self-adjoint for any eigenvalue $z_j \in \sigma_p(\mathcal{L})$ with $\operatorname{Im} z_j \neq 0$, while the operator $(\Omega_0 - z_j)^{-1} \sigma_3 P_{z_j}$ is real-valued for any $z_j \in \sigma_p(\mathcal{L})$ with $\operatorname{Im} z_j = 0$. On the other hand, the projection P_c to the non-singular essential spectrum $X_c(\mathcal{L})$ affects the imaginary part of (5.23) as follows:

$$\begin{aligned} \operatorname{Im} \hat{\Delta}_{i,j}(\epsilon) &= -\epsilon^2 \frac{\operatorname{Im} \langle P_c (\mathcal{L} - \Omega_0)^{-1} B_1^* A_1 \boldsymbol{\phi}_i, A_1^* B_1 \sigma_3 \boldsymbol{\phi}_j \rangle}{\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle} + \mathrm{O}(\epsilon^3) \\ &= -\epsilon^2 \frac{\operatorname{Im} \langle P_c (\mathcal{L} - \Omega_0)^{-1} V_1 \boldsymbol{\phi}_i, \sigma_3 V_1 \boldsymbol{\phi}_j \rangle}{\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle} + \mathrm{O}(\epsilon^3), \end{aligned} \tag{5.25}$$

where we have used that $V_1^* \sigma_3 = \sigma_3 V_1$. We show that the matrix with elements

$$\mathrm{Im}\langle P_c(\mathcal{L} - \Omega_0)^{-1} V_1 \boldsymbol{\phi}_i, \sigma_3 V_1 \boldsymbol{\phi}_j \rangle$$

is non-negative. We use wave operators and introduce the set $\{\hat{\boldsymbol{\phi}}_j\}_{j=1}^N$, such that $P_c V_1 \boldsymbol{\phi}_j = W \hat{\boldsymbol{\phi}}_j$. Using (4.16), we show that

$$\mathrm{Im}\langle P_c(\mathcal{L} - \Omega_0)^{-1} V_1 \boldsymbol{\phi}_i, \sigma_3 V_1 \boldsymbol{\phi}_j \rangle = \mathrm{Im}\langle (\mathcal{L} - \Omega_0)^{-1} W \hat{\boldsymbol{\phi}}_i, \sigma_3 W \hat{\boldsymbol{\phi}}_j \rangle$$
$$= \pi \langle \delta(\mathcal{L}_0 - \Omega_0) \hat{\boldsymbol{\phi}}_i, W^* \sigma_3 W \hat{\boldsymbol{\phi}}_j \rangle = \pi \langle \delta(\mathcal{L}_0 - \Omega_0) \hat{\boldsymbol{\phi}}_i, \sigma_3 \hat{\boldsymbol{\phi}}_j \rangle = \pi \left(\delta(-\Delta + \omega - \Omega_0) \hat{\phi}_{i1}, \hat{\phi}_{j1} \right). \quad (5.26)$$

The last matrix in (5.26) is non-negative. For a generic potential V_1, the matrix is strictly positive. It follows from (5.25) that the sign of eigenvalues of $\mathrm{Im}\,\hat{\Delta}$ is given by the signatures of $\langle \boldsymbol{\phi}_j, \sigma_3 \boldsymbol{\phi}_j \rangle = \Omega_0^{-1} \langle \boldsymbol{\phi}_j, \mathcal{H} \boldsymbol{\phi}_j \rangle$, $1 \le j \le N$. According to conditions of Proposition 5.1, there are k eigenvalues $z_j(\epsilon)$ of the matrix $\hat{\Delta}_{i,j}(\epsilon)$, such that $\mathrm{Im}\, z_j(\epsilon) > 0$ for $\epsilon \neq 0$ and $N - k$ eigenvalues $z_j(\epsilon)$ such that $\mathrm{Im}\, z_j(\epsilon) < 0$ for $\epsilon \neq 0$. The first k eigenvalues are true eigenvalues in the upper half-plane of z, while the other $N - k$ "eigenvalues" are resonances off the essential spectrum. No embedded eigenvalues exist near $z = \Omega_0$ for $\epsilon \neq 0$. ■

6 Bounds on the number of unstable eigenvalues

We conclude the paper with more precise statements on the number and type of unstable eigenvalues z with $\mathrm{Im}(z) > 0$ in the linearized NLS problem (1.2). There are two types of unstable eigenvalues: positive imaginary eigenvalues z, the number of which is denoted as N_{imag}, and complex eigenvalues in the first open quadrant, the number of which is denoted as N_{comp}, counting their multiplicity. The bounds on the number of unstable eigenvalues N_{imag} and N_{comp} were derived in [Pel03] with the use of Sylvester's Inertia Law of matrix analysis, in the context of the matrix linearized NLS problem. We show here that these bounds follow naturally from our main results, once the main Theorems 1, 2, and 3 are rewritten for operators $\hat{\mathcal{L}}_\pm$, defined by

$$\hat{\mathcal{L}}_+ = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\mathcal{L}}_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}.$$

It is clear from (3.2) that $n(\mathcal{H}) = n(\hat{\mathcal{L}}_+) + n(\hat{\mathcal{L}}_-)$, where $n(\hat{\mathcal{L}}_\pm)$ are the numbers of negative eigenvalues of $\hat{\mathcal{L}}_\pm$ in $L^2(\mathbb{R}^3, \mathbb{C}^2)$, counting their multiplicity. The bounds on the number of unstable eigenvalues are formulated as the corollary of the three propositions below.

Proposition 6.1 *Let Assumption 2.7 be satisfied, such that $Q'(\omega) \neq 0$, where $Q(\omega) = \int_{\mathbb{R}^3} \phi^2(x) dx$. Let $\hat{X}_0(\mathcal{L})$ be the constrained subspace of $L^2(\mathbb{R}^3, \mathbb{C}^2)$, defined in (3.16). Then $n(\hat{\mathcal{L}}_+)|_{\hat{X}_0} = n(\hat{\mathcal{L}}_+) - 1$ if $Q'(\omega) > 0$ and $n(\hat{\mathcal{L}}_+)|_{\hat{X}_0} = n(\hat{\mathcal{L}}_+)$ if $Q'(\omega) < 0$, while $n(\hat{\mathcal{L}}_-)|_{\hat{X}_0} = n(\hat{\mathcal{L}}_-)$ in either case.*

Proof. The statement follows from the proof of Theorem 1 in Section 3, see (3.35). ∎

We simplify the following proposition with an additional assumption.

Assumption 6.2 *No purely imaginary eigenvalues z of $\mathcal{L}$ exist, such that $(u, \mathcal{L}_+ u) = -(w, \mathcal{L}_- w) = 0$, where $(u, w)^T$ is the eigenvector of the problem (3.1).*

Proposition 6.3 *Let Assumptions 2.8 and 6.2 be satisfied. Let $N^-_{\rm real}$ and $N^+_{\rm real}$ be the number of positive real eigenvalues of $\mathcal{L}$ corresponding to eigenvectors $(u, w)^T$ with negative and positive values of $(u, \mathcal{L}_+ u)$, where $(u, \mathcal{L}_+ u) = (w, \mathcal{L}_- w)$. Let $N^-_{\rm imag}$ and $N^+_{\rm imag}$ be the number of positive purely imaginary eigenvalues of $\mathcal{L}$ corresponding to eigenvectors $(u, w)^T$ with negative and positive values of $(u, \mathcal{L}_+ u)$, where $(u, \mathcal{L}_+ u) = -(w, \mathcal{L}_- w)$. Let $\hat{X}_c(\mathcal{L})$ be the non-singular part of the essential spectrum of $\mathcal{L}$, defined in (3.17). Then*

$$
\begin{aligned}
n(\hat{\mathcal{L}}_+)|_{\hat{X}_c} &= n(\hat{\mathcal{L}}_+)|_{\hat{X}_0} - N^-_{\rm real} - N^-_{\rm imag} - N_{\rm comp}, && (6.1)\\
n(\hat{\mathcal{L}}_-)|_{\hat{X}_c} &= n(\hat{\mathcal{L}}_-) - N^-_{\rm real} - N^+_{\rm imag} - N_{\rm comp}. && (6.2)
\end{aligned}
$$

Proof. In the case of semi-simple eigenvalues, the statement follows from the proof of Theorem 2 in Section 3, see (3.38). Only multiple complex eigenvalues are allowed by Assumptions 2.8) and 6.2, but they do not change the reduction formulas (6.1) and (6.2). ∎

Proposition 6.4 *Let Assumptions 2.1, 2.6, and 2.8 be satisfied. The quadratic forms $(u, \mathcal{L}_+ u)$ and $(w, \mathcal{L}_- w)$ are strictly positive in $(u, w) \in \hat{X}_c(\mathcal{L})$.*

Proof. We use completeness of the space $\hat{X}_c(\mathcal{L})$, proved with wave operators in Proposition 4.1. Let $\{(u_\Omega, w_\Omega)^T\}_{\Omega \in \sigma_e(\mathcal{L})}$ be the basis of eigenfunctions in $\hat{X}_c(\mathcal{L})$. The continuous set of eigenfunctions is orthogonal with respect to the Dirac measure as

$$
(u_{\Omega'}, w_\Omega) = \rho_\Omega \delta(\Omega - \Omega'). \tag{6.3}
$$

Using the decomposition for $(u, w)^T \in \hat{X}_c(\mathcal{L})$:

$$
u(x) = \int_{\Omega \in \sigma_e(\mathcal{L})} a_\Omega u_\Omega(x) d\Omega, \qquad w(x) = \int_{\Omega \in \sigma_e(\mathcal{L})} b_\Omega w_\Omega(x) d\Omega, \tag{6.4}
$$

and the orthogonality relations (6.3), we represent the quadratic forms $(u, \mathcal{L}_+ u)$ and $(w, \mathcal{L}_- w)$ as follows:

$$
\begin{aligned}
(u, \mathcal{L}_+ u) &= \int_{\Omega \in \sigma_e(\mathcal{L})} \Omega \rho_\Omega |a_\Omega|^2 d\Omega, && (6.5)\\
(w, \mathcal{L}_- w) &= \int_{\Omega \in \sigma_e(\mathcal{L})} \Omega \rho_\Omega |b_\Omega|^2 d\Omega. && (6.6)
\end{aligned}
$$

Since h is positive definite in $\hat{X}_c(\mathcal{L})$, proved in Theorem 3, we conclude that $\rho_\Omega \in \mathbb{R}$ and $\rho_\Omega > 0$, for all $\Omega \in \sigma_c(\mathcal{L})$. Therefore, the quadratic forms in (6.5) and (6.6) are both positive definite. ∎

Corollary 6.5 *Let Assumptions 2.1, 2.6, 2.7, 2.8, and 6.2 be satisfied. The linearized NLS problem (1.2) has* $N_{\rm unst} = N_{\rm imag} + 2N_{\rm comp}$ *unstable eigenvalues* z *with* $\mathrm{Im}(z) > 0$*, where*

$$(i) \quad |n(\hat{\mathcal{L}}_+) - n(\hat{\mathcal{L}}_-) - p(Q')| \leq N_{\rm unst} \leq n(\hat{\mathcal{L}}_+) + n(\hat{\mathcal{L}}_-) - p(Q'), \tag{6.7}$$

$$(ii) \quad N_{\rm imag} \geq |n(\hat{\mathcal{L}}_+) - n(\hat{\mathcal{L}}_-) - p(Q')|, \tag{6.8}$$

$$(iii) \quad N_{\rm comp} \leq \min(n(\hat{\mathcal{L}}_+) - p(Q'), n(\hat{\mathcal{L}}_-)), \tag{6.9}$$

where $p(Q') = 1$ *if* $Q'(\omega) > 0$ *and* $p(Q') < 0$ *if* $Q'(\omega) < 0$.

Stability theorems of Grillakis, Shatah, & Strauss [GSS87, GSS90], Grillakis [G90], and Jones [J88] follow from Corollary 6.5. In particular, when $n(\hat{\mathcal{L}}_+) + n(\hat{\mathcal{L}}_-) - p(Q')$ is odd, then $|n(\hat{\mathcal{L}}_+) - n(\hat{\mathcal{L}}_-) - p(Q')|$ is odd, and there exists at least one unstable eigenvalue, $N_{\rm imag} \geq 1$ [GSS87, GSS90]. When $n(\hat{\mathcal{L}}_-) = 0$, all unstable eigenvalues are purely imaginary, such that $N_{\rm comp} = 0$ and $N_{\rm imag} = n(\hat{\mathcal{L}}_+) - p(Q')$ [J88, G90]. The case $n(\hat{\mathcal{L}}_-) = 0$ commonly occur for positive ground states of the NLS equation (1.3) [TY02a]. The case $n(\hat{\mathcal{L}}_-) = n(\hat{\mathcal{L}}_+) - p(Q')$ may occur for excited states of the NLS equation, when complex unstable eigenvalues are also possible [TY03b].

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