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On the 'vorticity' formulation of the adjoint equations and its solution using the vortex method†

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Abstract. Adjoint equations arise in many problems of optimal flow control and estimation. This paper discusses the 'vorticity' (i.e. non-primitive) formulation of such equations in the case of the wake control problem. First we derive the 'vorticity' form from the primitive form of the adjoint equations, and then show how it can be efficiently solved using the vortex method. We also discuss some implementation issues and present sample results.

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Contents

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1. Introduction

In this paper we study the non-primitive formulation of the momentum adjoint equations arising in flow control problems solved using optimal control theory methods (see [1] for a review). The particular problem we address here is rotary wake control for drag reduction in the laminar regime. Wakes are open flow systems characterized by concentrated vorticity distribution, consequently computation of such flow using vorticity (i.e. non-primitive) formulations of the Navier–Stokes system appears particularly attractive. In this paper we focus on some computational issues arising when one is using the vortex method to solve both the Navier–Stokes system and the associated adjoint equations resulting from the optimization problem specified in what follows. Here we consider 2D flows of viscous incompressible liquids governed by the Navier–Stokes equations (1) and taking place in an unbounded domain Ω. For the purpose of solving this equation with the use of the vortex method we also express it here in the vorticity form (2)

$$\begin{cases} \dfrac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = -\nabla p + \mu\Delta\vec{V}, \\ \nabla\cdot\vec{V} = 0, \\ \vec{V}|_{t=0} = \vec{V}_0 \qquad \text{in } \Omega, \\ \vec{V} = \vec{b}(\dot{\varphi}) \qquad \text{on } \Gamma_0, \\ \vec{V} \to \vec{V}_\infty \qquad \text{for } |x| \to \infty, \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + (\vec{V}\cdot\nabla)\omega = \mu\Delta\omega, \\ \omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}, \qquad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ \vec{V}|_{t=0} = \vec{V}_0 \qquad \text{in } \Omega, \\ \vec{V} = \vec{b}(\dot{\varphi}) \qquad \text{on } \Gamma_0, \\ \vec{V} \to \vec{V}_\infty \qquad \text{for } |x| \to \infty, \end{cases} \tag{2}$$

where $\vec{V} = [u, v]$ is the velocity field, ω is the vorticity, p is the pressure and μ denotes viscosity. The problems are supplemented with the initial condition $\vec{V}_0$, the boundary condition $\vec{b}(\dot{\varphi})$ ($\dot{\varphi}$ is the rotation rate of the obstacle) and the velocity at infinity $\vec{V}_\infty$. We remark here that the initial and boundary conditions for vorticity are also expressed in terms of velocity. Under certain technical assumptions (cf [2]), the two formulations (1) and (2) can be shown to be equivalent.

2. Optimal control algorithm

In this investigation we are interested in using the rotary motion of the circular obstacle to obtain drag reduction, while keeping the control effort as low as possible. Here we use the mathematical framework laid out in the seminal work of Abergel and Temam [3]. We also build on the results of Bewley *et al* [4] obtained using a similar approach applied to the channel flow problem. The goal stated above can be expressed as minimization of the following cost functional depending on the rotation rate $\dot{\varphi}$ as the control variable

$$\begin{aligned} J(\dot{\varphi}) &= \tfrac{1}{2}\int_0^T \left\{ \begin{bmatrix} \text{power related to} \\ \text{the drag force} \end{bmatrix} + \begin{bmatrix} \text{power needed to} \\ \text{control the flow} \end{bmatrix} \right\} \mathrm{d}t \\ &= \tfrac{1}{2}\int_0^T \oint_{\Gamma_0} \{[p(\dot{\varphi})\vec{n} - \mu\vec{n}\cdot \bar{\bar{D}}(\vec{V}(\dot{\varphi}))]\cdot[\dot{\varphi}(\vec{e}_z\times\vec{r}) + \vec{V}_\infty]\}\,\mathrm{d}\sigma\,\mathrm{d}t, \end{aligned} \tag{3}$$

where $\bar{\bar{D}}(\vec{V}) = [\nabla\vec{V} + (\nabla\vec{V})^T]$ and $\vec{e}_z$ is the versor of the Z-axis (perpendicular to the plane of motion). The minimum of the functional is characterized by the vanishing of its Gâteaux differential $J'(\dot{\varphi};h)$ for all arbitrarily chosen h. The Gâteaux differential is defined as $J'(\dot{\varphi};h) = \lim_{\epsilon\to 0}[J(\dot{\varphi}+\epsilon h) - J(\dot{\varphi})]/\epsilon$, where h represents the direction in which the control is perturbed. In the actual optimization the expression for $J'(\dot{\varphi};h)$ is used to extract the functional gradient ∇J according to $J'(\dot{\varphi};h) = (\nabla J, h)_{L_2([0,T])}$. The gradient of the functional obtained in this way can now be used in a conjugate gradient algorithm to iteratively find the minimizer starting from some initial guess for the control variable (taken here as $\dot{\varphi}_0 \equiv 0$). Due to space limitations, we are not able to present the complete derivation here and have to restrict ourselves to stating the main results only. The reader is referred to the works of Protas [5] and Protas and Styczek [6] for further details. The adjoint calculus is used to obtain a convenient expression for the functional gradient

$$\nabla J(t) = \tfrac{1}{2}\mu R^2 \int_0^{2\pi} [(s_{12}^0 + s_{12}^*)\cos(2\theta) - (s_{11}^0 + s_{11}^*)(\sin(2\theta))]\,\mathrm{d}\theta, \tag{4}$$

where s_{11}^0, s_{12}^0, s_{11}^* and s_{12}^* are the components of the rate-of-strain tensor for the primal and adjoint fields, $\vec{V}_0$ and $\vec{w}^*$, respectively. The adjoint velocity and pressure $\{\vec{w}^*, q^*\}$ solve the following *adjoint system*:

$$\begin{cases} N^*\vec{w}^* = \begin{bmatrix} -\frac{\partial\vec{w}^*}{\partial t} - \vec{V}_0\cdot[\nabla\vec{w}^* + (\nabla\vec{w}^*)^T] - \mu\Delta\vec{w}^* + \nabla q^* \\ -\nabla\cdot\vec{w}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vec{w}^*|_{t=T} = 0 \qquad \text{in } \Omega, \\ \vec{w}^* = \vec{r}\times(\dot{\varphi}\vec{e}_z) + \vec{V}_\infty \qquad \text{on } \Gamma_0, \qquad \vec{w}^*\to 0 \qquad \text{for } |x|\to\infty. \end{cases} \tag{5}$$

The primal field $\vec{V}_0$ is the state around which linearization is performed when computing the Gâteaux differential. Evidently, the adjoint system (5) is forced on the boundary by the quantity 'measured' in the linearized version of functional (3) (i.e. its Gâteaux differential). On the other hand, the quantities derived from the adjoint field $\vec{w}^*$ and represented in the gradient (4) give information about the sensitivity of functional (3) to the rotary control $\dot{\varphi}(t)$. The adjoint system (5) is a terminal value problem and has to be marched backwards in time. Using the substitution $t = T - \tau$ we, however, obtain a well-posed initial value problem in the time variable τ. In order to obtain the functional gradient $\nabla J(t)$ at every iteration of the conjugate gradient algorithm, we first solve the primal system (1) forward in time, then based on that, the adjoint system (5) backward in time and finally evaluate expression (4). Given the gradient, we perform line minimization of the functional to determine the optimal amplitude of the control. When the primal problem is solved in the vorticity form (2) using the vortex method, it is natural to use a similar approach to solve the adjoint system (5). In order to do this, below we derive the corresponding 'vorticity'† form.

† We use the term 'vorticity' in quotes, because this quantity is the curl of a field which is not, strictly speaking, a velocity field.

3. 'Vorticity' form of the adjoint equations

As is the case with the vorticity system (2), the 'vorticity' form of the adjoint system (5) is obtained by applying the curl operator to the first equation in (5). We define the adjoint 'vorticity' as $\vec{\omega}^* = \nabla \times \vec{w}^*$ and note that the curl operator commutes with the time derivative and the Laplacian, whereas the pressure term vanishes altogether. For the 'advection' term we obtain

$$[\nabla \times \{\vec{V}_0 \cdot [\nabla \vec{w}^* + (\nabla \vec{w}^*)^T]\}]_m = \varepsilon_{mkj} \frac{\partial}{\partial x_k}\left[V_{0i}\left(\frac{\partial w_j^*}{\partial x_i} + \frac{\partial w_i^*}{\partial x_j}\right)\right]$$
$$= \underbrace{V_{0i}\frac{\partial}{\partial x_i}\varepsilon_{mkj}\frac{\partial w_j^*}{\partial x_k}}_{(\vec{V}_0\cdot\nabla)\vec{\omega}^*} + \varepsilon_{mkj}\frac{\partial V_{0i}}{\partial x_k}\left(\frac{\partial w_j^*}{\partial x_i} + \frac{\partial w_i^*}{\partial x_j}\right). \tag{6}$$

Putting together these observations and denoting $\omega^* = \vec{\omega}^* \cdot \vec{e}_z$ we obtain in 2D (extension to 3D being straightforward) the 'vorticity' form of the adjoint system (5)

$$\begin{cases} -\dfrac{\partial \omega^*}{\partial t} - (\vec{V}_0 \cdot \nabla)\omega^* - \mu\Delta\omega^* + 2\left[\dfrac{\partial w_y^*}{\partial y}\left(\dfrac{\partial V_{0y}}{\partial x} + \dfrac{\partial V_{0x}}{\partial y}\right) + \dfrac{\partial V_{0x}}{\partial x}\left(\dfrac{\partial w_y^*}{\partial x} + \dfrac{\partial w_x^*}{\partial y}\right)\right] = 0, \\ \omega^* = \dfrac{\partial w_y^*}{\partial x} - \dfrac{\partial w_x^*}{\partial y}, \qquad \dfrac{\partial w_x^*}{\partial x} + \dfrac{\partial w_y^*}{\partial y} = 0, \\ \vec{w}^* = \vec{r} \times (\dot{\varphi}\vec{e}_z) + \vec{V}_\infty \quad \text{on } \Gamma_0, \qquad \vec{w}^* \to 0 \quad \text{for } |x| \to \infty, \\ \vec{w}^*|_{t=T} = 0 \quad \text{in } \Omega. \end{cases} \tag{7}$$

The source term (i.e. the last one) in the first equation above can be expressed using the components of the rate-of-strain tensors of the primal and the adjoint fields as $s_{11}^0 s_{12}^* - s_{11}^* s_{12}^0$. The above system has the form of an advection–diffusion equation with a source term. In this sense it resembles a linearization of the 2D vorticity system (2) in which the velocity is decoupled from vorticity and which is supplemented with a source term mentioned above. As was the case with the vorticity system (2), the boundary conditions are given in terms of the primitive adjoint variable and are unchanged with respect to the primitive form of the adjoint system. The meaning of the adjoint 'vorticity' ω^* is consistent with the definition of the adjoint state $\vec{w}^*$: it is used to evaluate expression (4) for the functional gradient. This expression involves the adjoint 'strains' s_{11}^* and s_{12}^* which are obtained by differentiating the field $\vec{w}^*$ calculated from ω^* by inverting the curl operator. The velocity field $\vec{V}_0$ in (7) also affects the adjoint 'vorticity' through the source term involving the strains s_{11}^0 and s_{12}^0. We emphasize here that using the 'vorticity' form (7) of the adjoint system (5) does not change the problem that is solved. It only modifies the way in which the sensitivity of the functional is calculated.

4. Numerical solution of the primal and adjoint systems in the vorticity form

In this work both problems are solved with the use of the vortex method. In what follows we give a few details concerning the solution of the primal system, and in addition we show how this approach can be applied to the adjoint system (7) in an efficient manner. In both cases the vorticity field is represented as a superposition of particles (i.e. vortex blobs) with Gaussian cores. The particles move in their own induced velocity field complemented by some potential contribution which accounts for the free stream at infinity and the effect of the boundary conditions. The induced velocity is computed using the fast multipole method of Greengard and Rokhlin [7] in the implementation developed by Wald and Styczek [8]. The general idea is to compute only the closest interactions with the use of the exact formulae, whereas the far

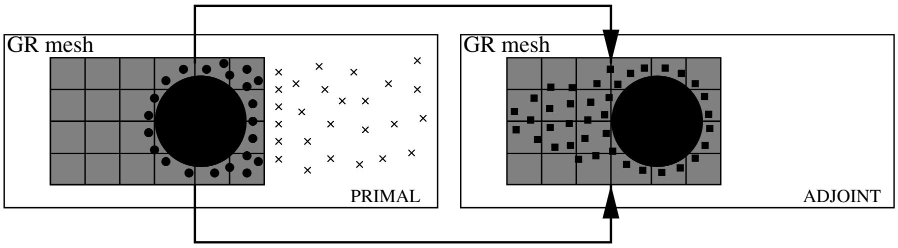

Figure 1. A schematic showing the transfer of information from the primal to the adjoint part of the code (see the text for explanations).

interactions are accounted for with the use of hierarchical series expansions. These expansions can be clustered in an adaptive fashion resulting in the operation count decrease from $O(N^2)$ to $O(N \log N)$. This approach is very flexible and also allows for an efficient computation of all velocity derivatives. Viscous diffusion is calculated using the modified redistribution method of Shankar and van Dommelen [9]. The velocity boundary conditions are enforced at every time step by creating on the boundary a vortex sheet that cancels the deviation of the boundary velocity from the prescribed value. The new circulation in then diffused to the particles in the flow field. The number of particles used in the simulations of the flows at $Re = 150$ was of the order $O(10^5)$. For further details concerning implementation and several benchmark tests of the method we refer the reader to the works [5] and [10].

The vortex method used to solve the adjoint problem is, in the main, the same as the approach outlined above; there are, however, two important novelties: the adjoint field evolves in the velocity field $\vec{V}_0$ obtained during the forward sweep of the primal system (2) and the vorticity form of the adjoint equations includes a source term that must be properly accounted for. The source term $s_{11}^0 s_{12}^* - s_{11}^* s_{12}^0$ represents the second, besides viscous diffusion, mechanism resulting in modifications of the strengths of the particles. When the particle representation is used, the strain fields s_{11} and s_{12} have essentially the same localizations as the vorticity field of the vortex blob. This, in particular, means that the source term practically vanishes away from the support of *both* the primitive and the adjoint vorticities ω and ω^*. As a result, new adjoint 'vorticity' due to the source term is created *only* by modifying the strengths of the existing particles, but not away from their support. The source term involves the strain components s_{11}^0 and s_{12}^0 computed by differentiating the primal velocity field $\vec{V}_0$. As already mentioned, advection of the adjoint 'vorticity' ω^* takes place in the reversed primal velocity field $\vec{V}_0$. Therefore, this field must be transferred to the adjoint part of the code without incurring prohibitive computational overhead or storage requirements. We accomplish this by transferring the Taylor series expansion coefficients used to represent the velocity field $\vec{V}_0$ on the finest level in the primal Greengard–Rokhlin algorithm. These coefficients have to be transferred only in the areas where the velocity field $\vec{V}_0$ will have to be evaluated in the adjoint part of the code, i.e. only from the finest level cells which contain adjoint 'vorticity' particles (the shaded areas in figure 1). Likewise, the data for the primal vorticity particles in these cells will also have to be transferred to the adjoint part of the code to allow for evaluation of near induction using exact formulae. In figure 1 such primal particles are marked as circles, whereas crosses represent the primal particles that do not have to be transferred. The adjoint particles are marked as squares. With this information available (expansion coefficients and particles), we can efficiently evaluate the complete primal velocity field $\vec{V}_0$ with its derivatives in the neighbourhoods of all adjoint 'vorticity' particles. Due to reversed time, the adjoint 'vorticity' field grows in the upstream direction and the supports of the primal and the adjoint vorticity overlap only in

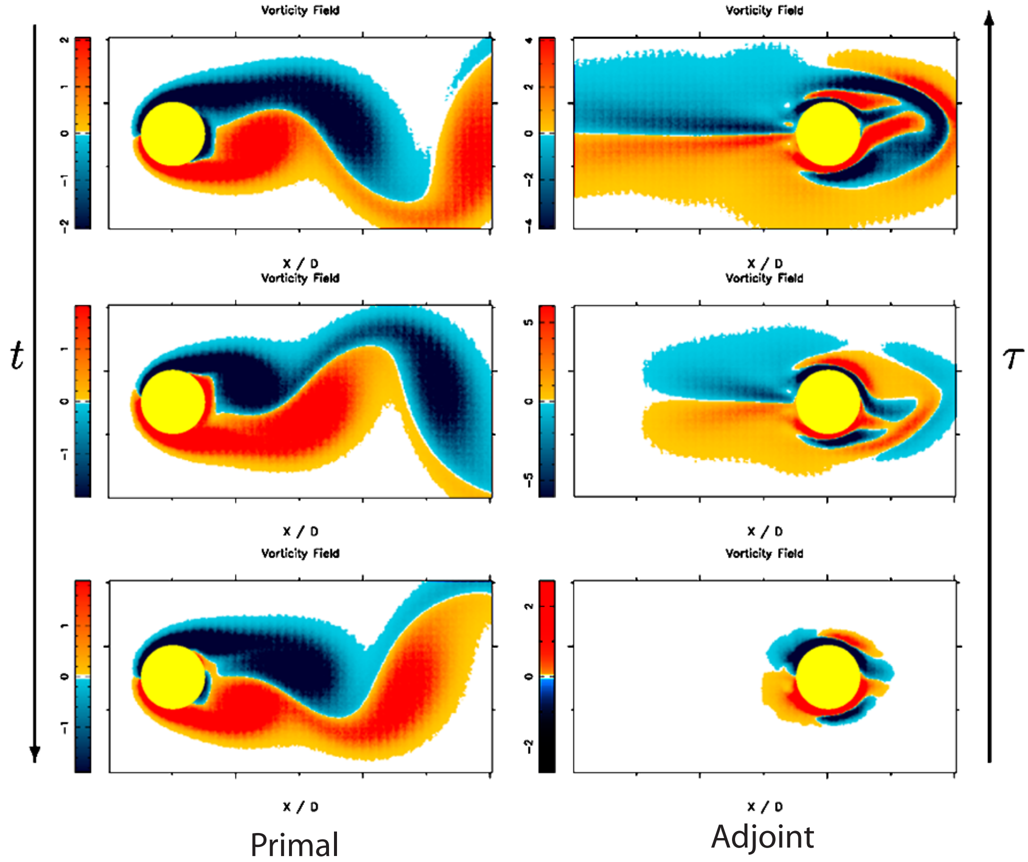

Figure 2. Fields of the primal (left) and the adjoint (right) vorticities at different stages during optimization. The arrows indicate the directions of the primal and the adjoint times, t and τ, respectively.

a small fraction of the computational domain. Consequently, using the expansion coefficients of the Greengard–Rokhlin algorithm, the amount of information transferred from the primal to the adjoint part of the code is relatively small and evaluation of the velocity $\vec{V}_0$ with its derivatives in the adjoint part of the code is very fast. We remark here that in the adjoint part of the code there is a separate Greengard–Rokhlin module handling the adjoint 'vorticity'. It is needed to evaluate the adjoint 'strains' in the source term and the adjoint 'velocity' on the boundary.

5. Sample results

In figure 2 we present the fields of the primal vorticity ω and the corresponding adjoint 'vorticity' ω^* obtained at different stages during the first iteration in the minimization of functional (3). Figure 3 (animation) presents the animated evolution of these two fields over the whole optimization interval $[0, T]$. These results correspond to the optimization horizon $T = 6$ and the Reynolds number $Re = 150$. Most interestingly, we remark that the adjoint field ω^* develops in the upstream direction and grows in the reversed time $\tau = T - t$, which is due to the fact that system (5), and therefore also (7), are terminal value problems. We recall that solutions of the primal and the adjoint systems are needed to determine the gradient of functional (3)

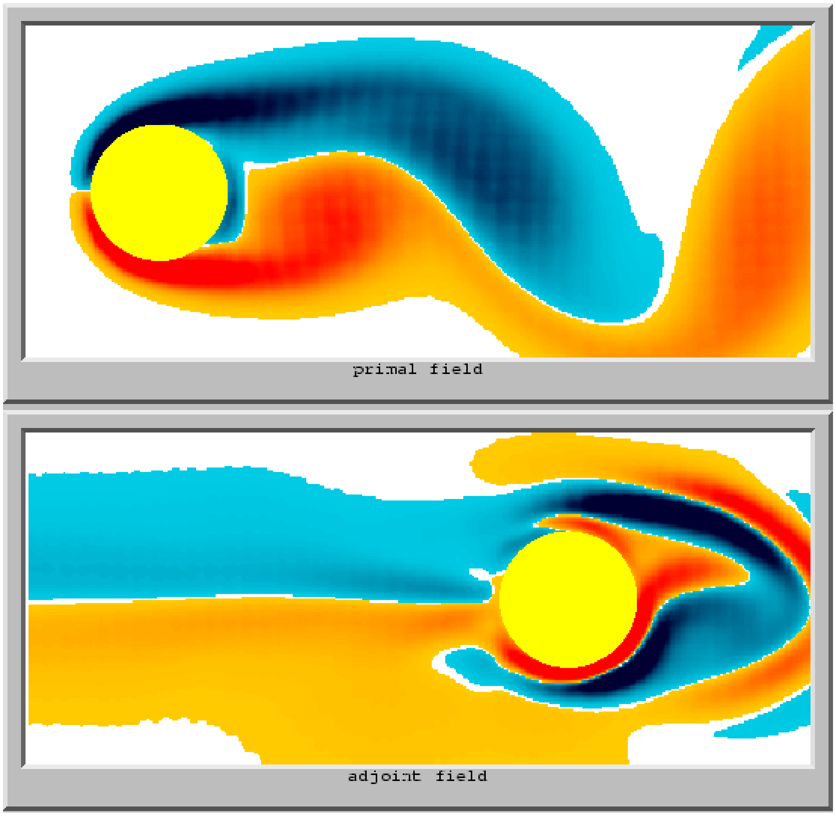

Figure 3. Animation showing the evolution of the primal vorticity field ω (top) and the corresponding adjoint field ω^* (bottom) over the optimization interval $[0, T]$.

with respect to control $\dot{\varphi}$ defined over the optimization interval $[0, T]$. The reader is referred to the works [5] and [6] for quantitative results concerning drag reduction and changes of the flow pattern in the controlled flows.

6. Conclusions

In this paper we derived and discussed some properties of the 'vorticity' (i.e. non-primitive) form of the adjoint equations arising in many optimization problems in hydrodynamics. We showed how the derived system can be solved using a generalization of the 2D vortex method, and how this approach can be efficiently implemented in the case when the vortex method is also used to solve the primal system. Some results were presented as regards the behaviour of the adjoint field obtained in optimization for drag reduction of the rotary control of the 2D cylinder wake flow.

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