Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations

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Abstract

Spectral stability analysis for solitary waves is developed in context of the Hamiltonian system of coupled nonlinear Schrödinger equations. The linear eigenvalue problem for a non-self-adjoint operator is studied with two self-adjoint matrix Schrödinger operators. Sharp bounds on the number and type of unstable eigenvalues in the spectral problem are found from inertia law for quadratic forms, associated with the two self-adjoint operators. Symmetry-breaking stability analysis is also developed with the same method.

Contents

1 Introduction

This paper addresses spectral stability of solitary waves in the system of N coupled nonlinear Schrödinger (NLS) equations,

$$i\frac{\partial\psi_n}{\partial z} + d_n\frac{\partial^2\psi_n}{\partial x^2} + f_n(|\psi_1|^2, ..., |\psi_N|^2)\psi_n = 0, \quad n = 1, .., N, \tag{1.1}$$

where $\psi_n(z,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$, $f_n : \mathbb{R}^N \to \mathbb{R}$, and $d_n \in \mathbb{R}$. We assume that $d_n > 0$, $f_n(0, ..., 0) = 0$, $n = 1, ..., N$, and

$$\frac{\partial f_n}{\partial|\psi_m|^2} = \frac{\partial f_m}{\partial|\psi_n|^2}, \quad n, m = 1, ..., N. \tag{1.2}$$

The system (1.1) has the following properties.

(i) The linear spectrum of (1.1) with $f_n \equiv 0$ is uncoupled,

$$\psi_n(z,x) = \int_{-\infty}^{\infty} \alpha_n(k_n) e^{i(k_n x + \omega_n(k_n) z)} dk_n, \quad \omega_n = -d_n k_n^2 \leq 0. \tag{1.3}$$

(ii) Any solution of (1.1) is invariant with respect to N phase rotations:

$$\psi_n(z,x) \mapsto e^{i\theta_n}\psi_n(z,x), \qquad \theta_n \in \mathbb{R}, \qquad n = 1, ..., N, \tag{1.4}$$

which are associated with N conserved charge functionals,

$$Q_n = \int_{\mathbb{R}} |\psi_n|^2 dx, \qquad \psi_n \in L^2(\mathbb{R}), \qquad n = 1, ..., N. \tag{1.5}$$

(iii) Any solution of (1.1) is invariant with respect to space translation:

$$\psi_n(z,x) \mapsto \psi_n(z, x - s), \qquad s \in \mathbb{R}. \tag{1.6}$$

(iv) Any solution of (1.1) is invariant with respect to Galileo translation:

$$\psi_n(z,x) \mapsto \psi_n(x - 2vz)\, e^{id_n^{-1}(vx - v^2 z)}, \qquad v \in \mathbb{R}. \tag{1.7}$$

(v) Under the condition (1.2), the system (1.1) conserves the Hamiltonian:

$$H = \int_{\mathbb{R}} \left[\sum_{n=1}^{N} d_n \left|\frac{\partial\psi_n}{\partial x}\right|^2 - U(|\psi_1|^2, ..., |\psi_N|^2)\right] dx, \qquad f_n = \frac{\partial U}{\partial|\psi_n|^2}, \tag{1.8}$$

and the momentum associated with the symmetry (1.6):

$$P = i\int_{\mathbb{R}} \left[\sum_{n=1}^{N} \left(\bar{\psi}_n \frac{\partial\psi_n}{\partial x} - \psi_n \frac{\partial\bar{\psi}_n}{\partial x}\right)\right] dx. \tag{1.9}$$

In this case, the system (1.1) takes the Hamiltonian form in canonical variables $\mathbf{u} = (u_1, ..., u_N)^T$ and $\mathbf{w} = (w_1, ..., w_N)^T$:

$$\frac{d}{dz}\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \frac{1}{2}\mathcal{J} H'(\mathbf{u}, \mathbf{w}), \qquad \mathcal{J} = \begin{pmatrix} \mathcal{O}_N & \mathcal{I}_N \\ -\mathcal{I}_N & \mathcal{O}_N \end{pmatrix}, \tag{1.10}$$

where $(\mathbf{u}, \mathbf{w})^T : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^{2N}$, $\mathcal{I}_N$ and $\mathcal{O}_N$ are identity and zero matrices in $\mathbb{R}^N$, $\mathcal{J}^+ = -\mathcal{J}$, and the Hamiltonian $H(\mathbf{u}, \mathbf{w})$ follows from (1.8) with $\psi_n = u_n + iw_n$ and $\bar{\psi}_n = u_n - iw_n$, $n = 1, ..., N$.

2 Main formalism

Stationary solutions of the coupled NLS equations (1.1) are defined by the standard ansatz:

$$\psi_n(z,x) = \Phi_n(x) e^{i\beta_n z}, \tag{2.1}$$

where $\Phi_n : \mathbb{R} \to \mathbb{R}$. Components $\Phi_n(x)$ satisfy the system of equations:

$$d_n \frac{d^2\Phi_n}{dx^2} - \beta_n \Phi_n + f_n(\Phi_1^2, ..., \Phi_N^2)\Phi_n = 0, \qquad \lim_{|x|\to\infty} \Phi_n(x) = 0. \tag{2.2}$$

Throughout the paper, we will assume that the existence problem has the following solution.

Assumption 2.1 *There exists an exponentially decaying solution $\mathbf{\Phi}(x) = (\Phi_1, ..., \Phi_N)^T \in \mathbb{R}^N$, $\mathbf{\Phi} \in L^2(\mathbb{R})$ in an open domain $\boldsymbol{\beta} = (\beta_1, ..., \beta_N)^T \in \mathcal{B} \subset \mathbb{R}^N$. The stationary solution is not degenerate, such that $\Phi_n(x) = 0$ only in a finite number of points $x \in \mathbb{R}$, $n = 1, ..., N$. The mapping $\boldsymbol{\beta} \to \mathbf{\Phi}(x)$ is C^1 on $\boldsymbol{\beta} \in \mathcal{B}$.*

Exponentially decaying solutions of (2.2) may exist only if $\beta_n > 0$ (assuming $d_n > 0$), $n = 1, ..., N$, when components $\Phi_n(x)$ decay asymptotically as

$$\lim_{x\to\pm\infty} |\Phi_n(x) e^{a_n|x|} - c_n^{\pm}| = 0, \qquad a_n = \sqrt{\frac{\beta_n}{d_n}} \; (> 0), \tag{2.3}$$

where $c_n^{\pm}$ are some non-zero constants. The constraint $\beta_n > 0$ is related to the constraint $\omega_n \leq 0$ in the linear spectrum (1.3). The spectrum of exponentially decaying stationary solutions (2.1) is isolated from the linear spectrum (1.3), when $\beta_n > 0$. Otherwise, as it happens for other systems of coupled NLS equations [PY02], the exponentially decaying solutions become embedded into the linear spectrum (embedded solitons). Such solutions are semi-stable due to nonlinearity-induced radiative decay, even if they are linearly stable [PY02]. We note that the algebraically decaying solutions may also exist in the system (2.2) for $\beta_n = 0$ and they are embedded into the edge of the linear spectrum at $\omega_n = 0$. We do not consider algebraically decaying solutions in this paper.

Definition 2.2 *Families of stationary solutions $\mathbf{\Phi}(x)$ are classified by the nodal index $\mathbf{i} = (i_1, ..., i_N)^T$, where i_n is the number of zeros of $\Phi_n(x)$ for $x \in \mathbb{R}$. The stationary solution $\mathbf{\Phi}(x)$ with $\mathbf{i} = \mathbf{0}$ is called the ground state.*

Lemma 2.3 *Stationary solutions $\mathbf{\Phi}(x)$ are critical points of the Lyapunov functional:*

$$\Lambda[\boldsymbol{\psi}] = H[\boldsymbol{\psi}] + \sum_{n=1}^{N} \beta_n Q_n[\boldsymbol{\psi}], \tag{2.4}$$

where Q_n and H are given by (1.5) and (1.8).

Proof. The first variation of $\Lambda[\boldsymbol{\psi}]$ vanishes if $\boldsymbol{\psi} = \boldsymbol{\Phi}(x)$ satisfies the system (2.2). ■

Definition 2.4 *Suppose the problem (2.2) has a stationary solution $\boldsymbol{\Phi}(x)$ in the parameter domain $\boldsymbol{\beta} \in \mathcal{B}$. Define the energy surface of the stationary solutions,*

$$\Lambda_s(\boldsymbol{\beta}) = H_s(\boldsymbol{\beta}) + \sum_{n=1}^{N} \beta_n Q_{ns}(\boldsymbol{\beta}), \tag{2.5}$$

where $H_s(\boldsymbol{\beta}) = H[\boldsymbol{\Phi}]$ and $Q_{ns}(\boldsymbol{\beta}) = Q_n[\boldsymbol{\Phi}]$. The Hessian matrix $\mathcal{U}$ of the energy surface $\Lambda_s(\beta)$ is a symmetric matrix with the elements

$$\mathcal{U}_{n,m} = \frac{\partial^2 \Lambda_s}{\partial \beta_n \partial \beta_m}. \tag{2.6}$$

Lemma 2.5 *Matrix elements of the Hessian matrix $\mathcal{U}$ are continuous functions of $\boldsymbol{\beta}$ in $\mathcal{B}$, computed as*

$$\mathcal{U}_{n,m} = \frac{\partial Q_{ns}}{\partial \beta_m} = 2\langle \Phi_n \mathbf{e}_n, \frac{\partial \boldsymbol{\Phi}}{\partial \beta_m} \rangle. \tag{2.7}$$

Matrix $\mathcal{U}$ in a domain $\boldsymbol{\beta} \in \mathcal{B}$ has N real bounded eigenvalues.

Proof. It follows from Lemma 2.3 that

$$\frac{\partial \Lambda_s}{\partial \beta_n} = Q_{ns} + \left(\frac{\partial H_s}{\partial \beta_n} + \sum_{m=1}^{N} \beta_m \frac{\partial Q_{ms}}{\partial \beta_n} \right) = Q_{ns}. \tag{2.8}$$

If $\boldsymbol{\Phi} \in L^2(\mathbb{R})$ and $\boldsymbol{\Phi}(x)$ is C^1 function of $\boldsymbol{\beta}$ in $\mathcal{B}$, the second derivatives of $\Lambda_s(\boldsymbol{\beta})$ exists and equal to (2.7). Since $\mathcal{U}$ is a symmetric matrix with bounded elements, all eigenvalues of $\mathcal{U}$ are real and bounded. ■

Definition 2.6 *Denote the number of negative, zero and positive eigenvalues of $\mathcal{U}$ as $n(\mathcal{U})$, $z(\mathcal{U})$, and $p(\mathcal{U})$, respectively, such that $n(\mathcal{U}) + z(\mathcal{U}) + p(\mathcal{U}) = N$.*

Linearization at the stationary solutions (2.1) is defined by the expansion,

$$\psi_n(z, x) = [\Phi_n(x) + U_n(z, x) + iW_n(z, x)]\, e^{i\beta_n z}, \tag{2.9}$$

where $(U_n, W_n)^T \in \mathbb{R}^2$ are perturbations functions. Neglecting nonlinear terms, we find that the perturbation vectors $\mathbf{U} = (U_1, ..., U_N)^T$ and $\mathbf{W} = (W_1, ..., W_N)^T$ satisfy the linearized system in Hamiltonian form:

$$\frac{d}{dz} \begin{pmatrix} \mathbf{U} \\ \mathbf{W} \end{pmatrix} = \mathcal{J} h'(\mathbf{U}, \mathbf{W}), \tag{2.10}$$

where the linearized Hamiltonian $h(\mathbf{U}, \mathbf{W})$ is the second variation of the Lyapunov functional (2.4):

$$h = \frac{1}{2} \delta^2 \Lambda = \langle \mathbf{U}, \mathcal{L}_1 \mathbf{U} \rangle + \langle \mathbf{W}, \mathcal{L}_0 \mathbf{W} \rangle, \tag{2.11}$$

and $\mathcal{L}_0$ and $\mathcal{L}_1$ are matrix Schrödinger operators with the elements:

$$(\mathcal{L}_0)_{n,m} = \left(-d_n \frac{d^2}{dx^2} + \beta_n - f_n(\Phi_1^2, ..., \Phi_N^2)\right) \delta_{n,m} \tag{2.12}$$

$$(\mathcal{L}_1)_{n,m} = \left(-d_n \frac{d^2}{dx^2} + \beta_n - f_n(\Phi_1^2, ..., \Phi_N^2)\right) \delta_{n,m} - 2 \frac{\partial f_n}{\partial \Phi_m^2} \Phi_n \Phi_m. \tag{2.13}$$

The diagonal operator $\mathcal{L}_0$ is a composition of N scalar Schrödinger operators. The matrix operator $\mathcal{L}_1$ is symmetric in the Hamiltonian case (1.2). Both quadratic forms in (2.11) are real-valued. The linearized problem (2.10) reduces to a linear eigenvalue problem after separation of variables: $\mathbf{U} = \mathbf{u}(x)e^{\lambda z}$, $\mathbf{W} = \mathbf{w}(x)e^{\lambda z}$. Eigenvalues λ are defined by the spectrum of the non-self-adjoint operator $\mathcal{A}$:

$$\mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{O}_N & \mathcal{L}_0 \\ -\mathcal{L}_1 & \mathcal{O}_N \end{pmatrix}. \tag{2.14}$$

The operator $\mathcal{A}$ is defined in $L^2(\mathbb{R} \mapsto \mathbb{C}^{2N})$, equipped with the inner product,

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}} \left(\sum_{n=1}^{N} \bar{f}_n(x) g_n(x) \right) dx. \tag{2.15}$$

We use standard definitions of eigenvalues of $\mathcal{A}$ from [HS96, Definition 1.4].

Definition 2.7 *The value λ is an eigenvalue of $\mathcal{A}$ if $\ker(\mathcal{A} - \lambda) \neq \{0\}$ in $L^2(\mathbb{R})$, such that there exists a non-zero vector function $(\mathbf{u}, \mathbf{w})^T \in \ker(\mathcal{A} - \lambda)$ called an eigenvector of $\mathcal{A}$. The dimension of $\ker(\mathcal{A} - \lambda)$ is called the geometric multiplicity of λ.*

Definition 2.8 *The discrete spectrum of $\mathcal{A}$, $\sigma_{\rm dis}(\mathcal{A})$ is the set of all eigenvalues of $\mathcal{A}$ with finite algebraic multiplicity which are isolated from the continuous spectrum of $\mathcal{A}$, $\sigma_{\rm con}(\mathcal{A})$. The embedded spectrum of $\mathcal{A}$, $\sigma_{\rm emb}(\mathcal{A})$ is the set of all eigenvalues with finite algebraic multiplicity which belong to the continuous spectrum of $\mathcal{A}$, including the boundary points. The essential spectrum of $\mathcal{A}$ is $\sigma_{\rm ess}(\mathcal{A}) = \sigma_{\rm con}(\mathcal{A}) \cup \sigma_{\rm emb}(\mathcal{A})$ and the point spectrum of $\mathcal{A}$ is $\sigma_p(\mathcal{A}) = \sigma_{\rm dis}(\mathcal{A}) \cup \sigma_{\rm emb}(\mathcal{A})$. The total spectrum of $\mathcal{A}$ is $\sigma(\mathcal{A}) = \sigma_{\rm dis}(\mathcal{A}) \cup \sigma_{\rm ess}(\mathcal{A}) = \sigma_{\rm p}(\mathcal{A}) \cup \sigma_{\rm con}(\mathcal{A})$.*

The continuous spectrum $\sigma_{\rm con}(\mathcal{A})$ may contain resonances, corresponding to bounded non-decaying eigenvectors, and semi-eigenvalues, corresponding to eigenvectors, which are decaying at one infinity and bounded at the other infinity. Definitions of resonances and semi-eigenvalues will be given in terms of the scattering matrix for the problem (2.14) (see Definition 7.1).

The non-self-adjoint linear eigenvalue problem (2.14) is formulated as a coupled system for two symmetric matrix Schrödinger operators $\mathcal{L}_0$ and $\mathcal{L}_1$. Spectrum of these operators is reviewed in the following statements.

Lemma 2.9 *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_0$ or $\mathcal{L}_1$. Continuous spectrum of $\mathcal{L}$ has N branches located at*

$$\sigma_{\rm con}(\mathcal{L}) = \cup_{1 \leq n \leq N} \{\lambda \in \mathbb{R} : \lambda \geq \beta_n\}. \tag{2.16}$$

Discrete and embedded spectrum of $\mathcal{L}$ has a finite number of eigenvalues located at

$$\begin{aligned} \sigma_{\rm dis}(\mathcal{L}) &= \cup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \lambda < \beta_{\min}\}, && (2.17)\\ \sigma_{\rm emb}(\mathcal{L}) &= \cup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \beta_{\min} \leq \lambda_m < \beta_{\max}\}, && (2.18)\end{aligned}$$

where $\beta_{\min} = \min_{1\leq n \leq N}(\beta_n)$ and $\beta_{\max} = \max_{1 \leq n \leq N}(\beta_n)$. The algebraic multiplicity of eigenvalues coincides with their geometric multiplicity and is at most N.

Proof. The matrix Schrödinger operator $\mathcal{L}$ has exponentially decaying potentials and becomes a diagonal differential operator in the limit $|x| \to \infty$. As a result, the continuous spectrum of $\mathcal{L}$ is defined by the Weyl's criterion and the point spectrum of $\mathcal{L}$ is finite-dimensional [HS96, Theorem 7.2]. Furthermore, since $\mathcal{L}$ is self-adjoint, the algebraic multiplicity of eigenvalues always coincides with their geometric multiplicity [HS96, Theorem 6.7].

Exponentially decaying solutions of the spectral problem $\mathcal{L}\mathbf{u} = \lambda\mathbf{u}$ are superposed in the limit $|x| \to \infty$ over a basis of N vector-functions $\mathbf{e}_n e^{-b_n|x|}$, $n = 1, ..., N$, where $b_n = \sqrt{(\beta_n - \lambda)/d_n}$. For $\lambda < \beta_{\min}$, all vector-functions are exponentially decaying and there exist no more than N linearly independent eigenvectors $\mathbf{u}(x)$ for some (isolated) values of λ. This contributes to eigenvalues of discrete spectrum (2.17). For $\lambda \geq \beta_{\max}$, all vector functions are non-decaying and no embedded eigenvalues may exist. For $\beta_{\min} \leq \lambda \leq \beta_{\max}$, some components $u_n(x)$ are decaying, while the other components $u_n(x)$ are non-decaying. Let N_1 be the number of non-decaying components. Then, there exist N_1 branches of the continuous spectrum of $\mathcal{L}$ at this value of λ, and an embedded eigenvalue (if it exists) corresponds to at most $N - N_1$ linearly independent decaying eigenvectors $\mathbf{u}(x)$. ∎

Lemma 2.10 *The kernel of $\mathcal{L}_0$ has a basis of N eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$. The kernel of $\mathcal{L}_1$ has at least one eigenvector $\mathbf{\Phi}'(x)$.*

Proof. The eigenvectors of the kernels of $\mathcal{L}_0$ and $\mathcal{L}_1$ are generated by the rotational and translational invariance (1.4) and (1.6), respectively. It follows from Lemma 2.9 that the eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ form a basis in the kernel of $\mathcal{L}_0$, when $\Phi_n(x) = 0$ only in a finite number of points $x \in \mathbb{R}$, $n = 1, ..., N$. ∎

Definition 2.11 *Denote the number of negative and zero eigenvalues of the discrete spectrum of operator $\mathcal{L}$ in $L^2(\mathbb{R})$ as $n(\mathcal{L})$ and $z(\mathcal{L})$, respectively. The Morse index for stationary solutions is*

$$n(h) = n(\mathcal{L}_1) + n(\mathcal{L}_0). \tag{2.19}$$

Lemma 2.12 *The negative index of $\mathcal{L}_0$ is*

$$n(\mathcal{L}_0) = \sum_{n=1}^{N} i_n, \tag{2.20}$$

where i_n is the number of zeros of $\Phi_n(x)$ for $x \in \mathbb{R}$. The negative index $n(\mathcal{L}_0)$ and the nodal index $\mathbf{i} = (i_1, ..., i_N)^T$ remain fixed in the open domain $\boldsymbol{\beta} \in \mathcal{B}$.

Proof. Since $\mathcal{L}_0$ is a diagonal composition of scalar Schrödinger operators, Sturm Oscillation Theorem applies. Each operator $(\mathcal{L}_0)_{n,n}$ has a zero bound state $\Phi_n(x)$, such that $n((\mathcal{L}_0)_{n,n}) = i_n$. Eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ form a basis in the kernel of $\mathcal{L}_0$, when $\Phi_n(x) = 0$ only in finite number of points $x \in \mathbb{R}$, and therefore, the index $n(\mathcal{L}_0)$ remains fixed for any continuous deformations of $\boldsymbol{\Phi}(x)$ in $\boldsymbol{\beta} \in \mathcal{B}$. ■

We finish this section with some general properties of the eigenvalue problem (2.14).

Lemma 2.13 *If λ is an eigenvalue of (2.14), so are $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$.*

Proof. This standard result for linear Hamiltonian systems follows from the fact that if $(\mathbf{u}, \mathbf{w})$ is the eigenvector of (2.14) with λ, then $(\mathbf{u}, -\mathbf{w})$, $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$, and $(\bar{\mathbf{u}}, -\bar{\mathbf{w}})$ are eigenvectors of (2.14) with $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$, respectively. ■

Definition 2.14 *The stationary solution (2.1) is spectrally unstable if there exists at least one eigenvalue λ such that* $\mathrm{Re}(\lambda) > 0$*. It is weakly spectrally stable if all eigenvalues λ are zero or purely imaginary.*

Spectral instability occurs when the eigenvalue problem (2.14) has a pair of real eigenvalues $(\lambda, -\lambda)$ or a quadruple of complex eigenvalues $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. Weak spectral stability does not yet guarantee strong spectral stability, since there may exist eigenvalues of higher algebraic multiplicity with $\mathrm{Re}(\lambda) = 0$, which lead to nonlinear instability of stationary solutions [CP03]. We shall study here the generic case, when no structurally unstable eigenvalues exist in the problem (2.14).

Assumption 2.15 *(i) The end points $\lambda = \pm i\beta_n$, $n = 1, ..., N$ are not resonances, (ii) $\sigma_{\mathrm{ess}}(\mathcal{A})$ does not include semi-eigenvalues or embedded eigenvalues, (iii) $\sigma_{\mathrm{dis}}(\mathcal{A})$ does not include non-zero eigenvalues of higher algebraic multiplicity, (iv) $z(\mathcal{L}_1) = 1$, and (v) $z(\mathcal{U}) = 0$.*

Bifurcations in the spectrum of $\mathcal{A}$ may occur when Assumption 2.15 is violated. Bifurcations in the eigenvalue problem (2.14) will be studied elsewhere.

Lemma 2.16 *Define the constrained function space* $X_c(\mathbb{R}) = X_c^{(u)} \oplus X_c^{(w)}$, *where*

$$X_c^{(u)} = \left\{ \mathbf{u} \in L^2(\mathbb{R}) : \langle \Phi_n \mathbf{e}_n, \mathbf{u} \rangle = 0, \ n = 1, ..., N \right\}, \tag{2.21}$$

$$X_c^{(w)} = \left\{ \mathbf{w} \in L^2(\mathbb{R}) : \langle \mathbf{\Phi}', \mathbf{w} \rangle = 0 \right\}. \tag{2.22}$$

Eigenvectors $(\mathbf{u}, \mathbf{w})^T$ *in the problem (2.14) for* $\lambda \neq 0$ *belongs to the space* $X_c(\mathbb{R})$.

Proof. The linear eigenvalue problem (2.14) is written as a coupled system:

$$\mathcal{L}_1 \mathbf{u} = -\lambda \mathbf{w}, \qquad \mathcal{L}_0 \mathbf{w} = \lambda \mathbf{u}. \tag{2.23}$$

The constraints in (2.21)–(2.22) follow from the Fredholm's Alternative Theorem applied to (2.23) for $\lambda \neq 0$ with the set of eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ of the kernel of $\mathcal{L}_0$ and with the eigenvector $\mathbf{\Phi}'(x)$ of the kernel of $\mathcal{L}_1$. ∎

Lemma 2.17 *When* $z(\mathcal{L}_1) = 1$ *and* $z(\mathcal{U}) = 0$, *the geometric multiplicity of the null eigenvalue of* $\mathcal{A}$ *is exactly* $(N+1)$ *and the algebraic multiplicity of the null eigenvalue of* $\mathcal{A}$ *is exactly* $(2N+2)$.

Proof. The null space of $\mathcal{A}$ is spanned by at least $(N+1)$ eigenvectors:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \left\{ \left\{ \begin{pmatrix} \mathbf{0}_N \\ \Phi_n(x)\mathbf{e}_n \end{pmatrix} \right\}_{n=1}^N, \begin{pmatrix} \mathbf{\Phi}'(x) \\ \mathbf{0}_N \end{pmatrix} \right\}. \tag{2.24}$$

The generalized null space of $\mathcal{A}$ includes at least $(N+1)$ generalized eigenvectors:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \left\{ \left\{ \begin{pmatrix} \frac{\partial \mathbf{\Phi}}{\partial \beta_n} \\ \mathbf{0}_N \end{pmatrix} \right\}_{n=1}^N, \begin{pmatrix} \mathbf{0}_N \\ -\frac{1}{2} x \mathcal{D}^{-1} \mathbf{\Phi}(x) \end{pmatrix} \right\}, \tag{2.25}$$

where $\mathcal{D}$ is a diagonal matrix of $(d_1, ..., d_N)$. It follows from Lemma 2.10 that the $(N+1)$ eigenvectors (2.24) form a basis for null space of $\mathcal{A}$, when $z(\mathcal{L}_1) = 1$. Fredholm's Alternative Theorem applied to the first N generalized eigenvectors in (2.25) fail, when $z(\mathcal{U}) = 0$, i.e. no second generalized eigenvectors exist, see [S00]. Fredholm's Alternative Theorem always fails for the last generalized eigenvector in (2.25) [S00]. ∎

Corollary 2.18 *When* $z(\mathcal{L}_1) = 1$ *and* $z(\mathcal{U}) = 0$, *the generalized eigenvectors (2.25) do not belong to the constrained space* $X_c(\mathbb{R})$, *defined by (2.21)–(2.22).*

3 Main results

Stability of solitary waves in nonlinear Schrödinger (NLS) equations was studied extensively in the recent past. The first stability-instability theorem for a scalar NLS equation (1.1) with $N = 1$

was proven by Shatah and Strauss [SS85] and Weinstein [W86]. Only positive stationary solutions (ground states) were considered in one, two, and three spatial dimensions. Ground states have the nodal index $i = 0$ and the Morse index $n(h) = 1$. A single negative eigenvalue of h does not necessary lead to spectral instability in the linearized problem (2.14) because of the constraints in (2.21)–(2.22). If $p(\mathcal{U}) = 0$, the stationary solution $\Phi(x)$ is spectrally unstable and the linearized problem (2.14) has a single real positive eigenvalue λ. If $p(\mathcal{U}) = 1$, the solitary wave is weakly spectrally stable and all eigenvalues λ are purely imaginary [SS85, W86].

More formal and general analysis was developed by Grillakis, Shatah and Strauss [GSS87, GSS90] by using multi-dimensional Lie groups and spectral decompositions. The following theorems were proven for an abstract Hamiltonian system with symmetries, which includes the system of coupled NLS equations (1.10).

Theorem 1 *[GSS90] Let $z(\mathcal{U}) = 0$, then $p(\mathcal{U}) \leq n(h)$. A stationary solution (2.1) is weakly spectrally stable if $n(h) = p(\mathcal{U})$ and it is spectrally unstable if $n(h) - p(\mathcal{U})$ is odd. The linearized problem (2.14) has at least one real positive eigenvalue λ if $n(h) - p(\mathcal{U})$ is odd.*

Theorem 2 *[GSS90] The linearized problem (2.14) has at most $n(h)$ unstable eigenvalues λ such that* $\mathrm{Re}(\lambda) > 0$.

Theorem 3 *[GSS90] The linearized Hamiltonian h in constrained space $X_c(\mathbb{R})$ has the negative index $\#_{<0}(h) = n(h) - p(\mathcal{U}) - z(\mathcal{U})$ and the null index $\#_{=0}(h) = z(h) + z(\mathcal{U})$.*

Theorem 1 is the main stability–instability theorem in [GSS90]. Theorem 2 is formulated in [GSS90, Theorem 5.8] for a quadrant: $\mathrm{Re}(\lambda) < 0$, $\mathrm{Im}(\lambda) > 0$. The method of the proof can however be applied to the left half-plane $\mathrm{Re}(\lambda) < 0$, or equivalently, to the right half-plane $\mathrm{Re}(\lambda) > 0$. Theorem 3 is formulated in [GSS90, Theorem 3.1] as a more general statement, which is equivalent to Theorem 3 under Assumption 2.1 ($z_0 = 0$ in notations of [GSS90]).

Theorem 1 generalizes stability-instability theory in finite-dimensional Hamiltonian systems with symmetries [M85]. Since the positive ground state (2.1) with $N = 1$ has always indices $n(\mathcal{L}_1) = 1$ and $n(\mathcal{L}_0) = 0$, its stability and instability is uniquely described by Theorem 1. However, many examples showed insufficiency of Theorem 1 for complete stability-instability analysis. For instance, a scalar NLS equation in two dimensions has radially symmetric multiple pulses with the nodal index $i > 0$ and the Morse index $n(h) \geq 1 + 2i$ [J88a]. When $p(\mathcal{U}) = 1$ and $n(h) - p(\mathcal{U}) \geq 2i$ is even, Theorem 1 can not be applied.

While a simple application of Theorem 1 to the case of multicomponent stationary solutions (2.1) with $N > 1$ is given in [GSS90, Theorem 9.1], we note that Theorem 9.1 in [GSS90] derives a scalar stability criterion, computed from the minimal value $\beta_{\min} = \min_{1 \leq n \leq N}(\beta_n)$. The scalar criterion generally fails for $N > 1$, as the Morse index $n(h)$ of the stationary solution (2.1) is assumed to be one in [GSS90], which does not generally hold for $N > 1$.

More special instability theorems were found by Jones [J88a, J88b] and Grillakis [G88, G90] for the scalar NLS equation (1.1) with $N = 1$. Jones [J88a, J88b] used topological and shooting methods of dynamical systems theory. When $n(\mathcal{L}_1) - p(\mathcal{U}) > n(\mathcal{L}_0)$, theorems in [J88a, J88b] predict an unstable eigenvalue, no matter whether $n(h)$ is odd or even. The results apply to instability of radially symmetric solutions with nodal index $i > 0$ in two spatial dimensions [J88a], as well as to stability–instability of symmetric and anti-symmetric solutions in the NLS equation with x-dependent nonlinear function $f = f(x; |\psi|^2)$ [J88b].

Theorem 4 *[J88a, J88b] The linearized problem (2.14) with $N = 1$ has a real positive eigenvalue λ if $|n(\mathcal{L}_1) - n(\mathcal{L}_0)| > 1$.*

Grillakis [G88, G90] used theory of linear operators, orthogonal projections and quadratic forms and proved some general results for the linearized problem (2.14). In this context, the problem (2.14) is reformulated as a generalized eigenvalue problem for operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$. When $n(\mathcal{L}_0) = 0$, unstable eigenvalues λ may occur only as real positive eigenvalues [G88]. When $n(h) - p(\mathcal{U}) > 1$ and $n(\mathcal{L}_0) \neq 0$, complex unstable eigenvalues λ may also occur in the linearized problem (2.14) [G90].

Theorem 5 *[G88] Let $\#_{<0}(\mathcal{L}_1)$ and $\#_{<0}(\mathcal{L}_0)$ be the negative indices of operators $\mathcal{L}_1$ and $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$. The linearized problem (2.14) has at least $|\#_{<0}(\mathcal{L}_1) - \#_{<0}(\mathcal{L}_0)|$ real positive eigenvalues λ. If $\#_{<0}(\mathcal{L}_0) = 0$, the linearized problem has exactly $\#_{<0}(\mathcal{L}_1)$ real positive eigenvalues λ.*

Theorem 5 is formulated in [G88, Theorem 1.2]. The theorem is more precise and general compared to Theorem 4, the latter takes the worst case, when $p(\mathcal{U}) = 1$ for $N = 1$. It remains unclear how Theorem 5, which exploits a special structure of the linearized problem (2.14), is related to general Theorem 1 for an abstract Hamiltonian system. It also remains unclear how the bounds on the number of unstable eigenvalues can be extended in the case of complex eigenvalues in the linearized problem (2.14).

We shall revisit here the problem of spectral stability of stationary solutions (2.1) in the coupled NLS equations (1.1) with $N \geq 1$. We develop two new methods of analysis: (i) negative eigenvalues of a constrained spectral problem are counted from matrix analysis and (ii) the negative subspace of a linear differential matrix operator with positive continuous spectrum is proved to be invariant in two diagonal representations. The first method develops the matrix variant of the Vakhitov–Kolokolov method, previously studied in [PK00]. The second method develops the Sylvester's Inertia Theorem for quadratic forms associated with finite-dimensional (matrix) operators [G61], applied to finite-dimensional Hamiltonian systems in [M88]. The new methods of analysis are used to prove the following main results.

Theorem 6 (Negative index of constrained operators) *Operator $\mathcal{L}_1$ in constrained space $X_c^{(u)}(\mathbb{R})$ has exactly $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$ negative eigenvalues and $\#_{=0}(\mathcal{L}_1) = z(\mathcal{L}_1) + z(\mathcal{U})$ zero eigenvalues. Operator $\mathcal{L}_0$ in the constrained space $X_c^{(u)}(\mathbb{R})$ has exactly $\#_{<0}(\mathcal{L}_0) = n(\mathcal{L}_0)$ negative eigenvalues.*

Corollary 3.1 *Let $z(\mathcal{U}) = 0$. Then, $p(\mathcal{U}) \leq n(\mathcal{L}_1)$.*

Theorem 7 (Closure relation for negative index) *Assume that Assumption 2.15 is satisfied. Let $N_{\rm real}$ be the number of real positive eigenvalues λ of the problem (2.14), $2N_{\rm comp}$ be the number of complex eigenvalues λ with $\mathrm{Re}(\lambda) > 0$, and $2N_{\rm imag}^-$ be the number of purely imaginary eigenvalues λ with $\langle \mathbf{u}, \mathcal{L}_1 \mathbf{u} \rangle = \langle \mathbf{w}, \mathcal{L}_0 \mathbf{w} \rangle < 0$. Dimension of the negative subspace of the linearized Hamiltonian h in $X_c(\mathbb{R})$ is invariant as*

$$\#_{<0}(h) = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0) = N_{\rm real} + 2N_{\rm comp} + 2N_{\rm imag}^-. \tag{3.1}$$

Theorem 8 (Bounds on unstable eigenvalues) *Assume that Assumption 2.15 is satisfied. The linearized problem (2.14) has $N_{\rm unst} = N_{\rm real} + 2N_{\rm comp}$ unstable eigenvalues λ with $\mathrm{Re}(\lambda) > 0$, such that*

$$(i) \quad |n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| \leq N_{\rm unst} \leq (n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)), \tag{3.2}$$

$$(ii) \quad N_{\rm real} \geq |n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)|, \tag{3.3}$$

$$(iii) \quad N_{\rm comp} \leq \min\left(n(\mathcal{L}_0), n(\mathcal{L}_1) - p(\mathcal{U})\right). \tag{3.4}$$

Corollary 3.2 *Let $z(\mathcal{U}) = 0$. When $n(\mathcal{L}_0) = 0$, the linearized problem (2.14) has exactly $N_{\rm real} = n(\mathcal{L}_1) - p(\mathcal{U})$ real positive eigenvalues λ. If both $n(\mathcal{L}_0) = 0$ and $n(\mathcal{L}_1) = p(\mathcal{U})$, the stationary solution (2.1) is weakly spectrally stable.*

Theorem 6 decomposes general Theorem 3 in the case, when h is a sum of two quadratic forms for $\mathcal{L}_1$ and $\mathcal{L}_0$ as in (2.11). As a result, the upper bound on $p(\mathcal{U})$ of Theorem 1 is improved as $p(\mathcal{U}) \leq n(\mathcal{L}_1) \leq n(h)$, as in Corollary 3.1. Also the stability criterion of Theorem 1 decomposes into two conditions: $n(\mathcal{L}_1) = p(\mathcal{U})$ and $n(\mathcal{L}_0) = 0$, as in Corollary 3.2.

Theorem 7 gives a precise statement of the closure relation between indices $n(\mathcal{L}_0)$, $n(\mathcal{L}_1)$ and $p(\mathcal{U})$ on one side and $N_{\rm real}$, $N_{\rm comp}$ and $N_{\rm imag}^-$ on the other side. This theorem generalizes earlier results for $\#_{<0}(\mathcal{L}_1) = \#_{<0}(\mathcal{L}_0)$ formulated in [G88, Theorem 1.3] (when $N_{\rm comp} = N_{\rm imag}^- = 0$) and in [G90, Theorem 2.3] (when $N_{\rm real} = N_{\rm imag}^- = 0$).

Theorem 8 is a corollary of Theorems 5, 6, and 7. The lower bound in (3.2) is identical to that in Theorem 5 in view of Theorem 6. The upper bound in (3.2) improves Theorem 2. Theorem 8 also agrees with the instability criterion of Theorem 1. Let $z(\mathcal{U}) = 0$ and $m = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)$ be odd. Then $|n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| = |m - 2n(\mathcal{L}_0)| > 0$ and $N_{\text{unst}} > 0$. Therefore, Theorem 8 also guarantees instability for odd m, as Theorem 1.

When $\mathcal{L}_1$ and $\mathcal{L}_0$ are finite-dimensional operators, Theorems 6, 7, and 8 reduce to stability-instability theorems for critical points in finite-dimensional Hamiltonian systems with symmetry constraints [M85, M88]. When $n(\mathcal{L}_0) = 0$, the quadratic form $\langle \mathbf{W}, \mathcal{L}_0 \mathbf{W} \rangle$ in (2.11) is equivalent to a positive-definite kinetic energy, while the quadratic form $\langle \mathbf{U}, \mathcal{L}_1 \mathbf{U} \rangle$ in (2.11) is equivalent to a sign-indefinite potential energy. The Morse index of $\mathcal{L}_1$ under constraints (2.21) is $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$. When $n(\mathcal{L}_0) = 0$, the Morse index defines uniquely the unstable subspace of the linearized system according to Corollary 3.2. When $n(\mathcal{L}_0) > 0$ and both $\mathcal{L}_0$ and $\mathcal{L}_1$ are not positive definite, complex instabilities may occur and they are defined by Theorems 7 and 8.

In the end of this section, we show that the constraint (2.22) does not appear in Theorems 6, 7, and 8, due to Galileo invariance (1.7). A general family of stationary solutions is defined as

$$\psi_n(z,x) = \Psi_n(x - 2vz - s) e^{i\omega_n z + i\theta_n}. \tag{3.5}$$

The general stationary solutions (3.5) are critical points of the Lyapunov functional in the form:

$$\Lambda[\boldsymbol{\psi}] = H[\boldsymbol{\psi}] + \sum_{n=1}^{N} \omega_n Q_n[\boldsymbol{\psi}] + v P[\boldsymbol{\psi}], \tag{3.6}$$

where Q_n and P are given by (1.5) and (1.9). The general Hessian matrix $\mathbf{U}_H$ has the structure:

$$\mathcal{U}_H = \begin{bmatrix} \mathcal{U} & \frac{\partial \mathbf{Q}_s}{\partial v} \\ \frac{\partial \mathbf{Q}_s^T}{\partial v} & \frac{\partial P_s}{\partial v} \end{bmatrix}, \tag{3.7}$$

where $\mathbf{Q}_s = (Q_{1s}, ..., Q_{Ns})^T$, $Q_{ns}(\boldsymbol{\omega}, v) = Q_n[\boldsymbol{\Psi}]$, and $P_s(\boldsymbol{\omega}, v) = P[\boldsymbol{\Psi}]$. It follows from Galileo invariance (1.7) that a transformation,

$$\Psi_n(x) = \Phi_n(x) e^{i d_n^{-1} v x}, \quad \omega_n = \beta_n + \frac{v^2}{d_n}, \tag{3.8}$$

expresses $Q_{ns}(\boldsymbol{\omega}, v)$, $P_s(\boldsymbol{\omega}, v)$ as functions of $\boldsymbol{\beta}$:

$$\frac{\partial Q_{ns}}{\partial v}(\boldsymbol{\omega}, v) = -\sum_{m=1}^{N} \frac{2v}{d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}), \tag{3.9}$$

$$\frac{\partial P_s}{\partial v}(\boldsymbol{\omega}, v) = -\sum_{n=1}^{N} \frac{2}{d_n} Q_{ns}(\boldsymbol{\beta}) + \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{4v^2}{d_n d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}). \tag{3.10}$$

As a result, the quadratic form for $\mathbf{x} \in \mathbb{R}^{N+1}$ transforms to a quadratic form for $\mathbf{y} \in \mathbb{R}^N$ as

$$\langle \mathbf{x}, \mathcal{U}_H \mathbf{x} \rangle_{\mathbb{R}^{N+1}} = \langle \mathbf{y}, \mathcal{U} \mathbf{y} \rangle_{\mathbb{R}^N} - \left(\sum_{n=1}^{N} \frac{2Q_{ns}}{d_n} \right) x_{N+1}^2, \tag{3.11}$$

where

$$y_n = x_n - \frac{2v}{d_n} x_{N+1}, \qquad n = 1, ..., N,$$

and

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}^N} = \sum_{n=1}^{N} a_n b_n. \tag{3.12}$$

The additional eigenvalue for x_{N+1} is always negative, such that $p(\mathcal{U}_H) = p(\mathcal{U})$ and $z(\mathcal{U}_H) = z(\mathcal{U})$. Therefore, stability theorems are not affected by the constraint (2.22), due to Galileo invariance (1.7).

4 Eigenvalues of constrained spectral problems for $\mathcal{L}_1$ and $\mathcal{L}_0$

Here we prove Theorem 6 by counting eigenvalues of constrained spectral problems for $\mathcal{L}_1$ and $\mathcal{L}_0$ from matrix analysis. This method was announced in [PK00]. Constrained spectral problems were also considered in [BP93, G88].

Given the spectrum of $\mathcal{L}_1$ and $\mathcal{L}_0$ in $L^2(\mathbb{R})$, we study the spectrum of operators $\mathcal{L}_1$ and $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$, defined by (2.21). The constrained space $X_c^{(u)}(\mathbb{R})$ is an orthogonal compliment of the kernel of $\mathcal{L}_0$ in $L^2(\mathbb{R})$. The spectrum of $\mathcal{L}_1$ and $\mathcal{L}_0$ is complete in $X_c^{(u)}(\mathbb{R})$, due to the abstract result in [HS96, Proposition 2.7].

Proposition 4.1 *[HS96] Let M be a closed subspace of a Hilbert space $\mathcal{H}$ and $M^\perp$ be the orthogonal complement of M in $\mathcal{H}$, such that $M^\perp = \{x \in \mathcal{H} : \langle x, m \rangle = 0 \ \ \forall m \in M\}$. The subset $M^\perp$ is a closed subspace of $\mathcal{H}$ and is therefore a Hilbert space.*

Proposition 4.2 *Let negative eigenvalues of $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$ be defined by the problem:*

$$\mathcal{L}_0 \mathbf{u} = \lambda \mathbf{u}, \qquad \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \qquad \lambda < 0. \tag{4.1}$$

Then, $\#_{<0}(\mathcal{L}_0) = n(\mathcal{L}_0)$.

Proof. Eigenvectors of $\mathcal{L}_0$ for negative eigenvalues λ are orthogonal to the eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ of the kernel of $\mathcal{L}_0$. Therefore, they belong to $X_c^{(u)}(\mathbb{R})$. ∎

Proposition 4.3 *Let $z(\mathcal{L}_1) = 1$. Let negative and zero eigenvalues of $\mathcal{L}_1$ in $X_c^{(u)}(\mathbb{R})$ be defined by the problem:*

$$\mathcal{L}_1 \mathbf{u} = \lambda \mathbf{u} - \sum_{m=1}^{N} \nu_m \Phi_m(x) \mathbf{e}_m, \qquad \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \qquad \lambda \le 0, \tag{4.2}$$

where $\nu_1, \ldots, \nu_N$ are Lagrange multipliers. Then, $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$ and $\#_{=0}(\mathcal{L}_0) = 1 + z(\mathcal{U})$.

In order to prove Proposition 4.3, we introduce some notations. We denote negative eigenvalues of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ as λ_{-k} with orthonormal eigenvectors $\mathbf{u}_{-k}(x)$, accounting their multiplicity. We order negative eigenvalues from the minimal eigenvalue $\lambda_{-n(\mathcal{L}_1)}$ to the maximal eigenvalue $\lambda_{-1} < 0$. We also write spectral decomposition in $L^2(\mathbb{R})$ as a sum of three terms $\sum_{\lambda_{-k}<0}$, $\sum_{\lambda_{-k}=0}$, and $\sum_{\lambda_{-k}>0}$, where $\sum_{\lambda_{-k}<0}$ denotes $n(\mathcal{L}_1)$ terms from negative discrete spectrum of $\mathcal{L}_1$, $\sum_{\lambda_{-k}=0}$ denotes $z(\mathcal{L}_1) = 1$ term from the kernel of $\mathcal{L}_1$, and $\sum_{\lambda_k>0}$ denotes positive spectrum of $\mathcal{L}_1$.

When $\ker(\mathcal{L}_1 - \lambda) = \{0\}$ in $L^2(\mathbb{R})$, the constrained spectral problem (4.2) has a solution only if there exists a non-zero solution of the homogeneous linear system for ν_1,...,ν_N:

$$\sum_{m=1}^{N} \langle \Phi_n \mathbf{e}_n, (\lambda - \mathcal{L}_1)^{-1} \Phi_m \mathbf{e}_m \rangle \nu_m = 0, \qquad n = 1, ..., N. \tag{4.3}$$

By spectral calculus [RS78], the linear system (4.3) is equivalent to zero eigenvalues of the matrix $\mathcal{A}(\lambda)$ with the elements,

$$\mathcal{A}_{n,m}(\lambda) = \sum_{\lambda_{-k}<0} \frac{\langle \Phi_n \mathbf{e}_n, \mathbf{u}_{-k}\rangle \langle \mathbf{u}_{-k}, \Phi_m \mathbf{e}_m\rangle}{\lambda - \lambda_{-k}} + \sum_{\lambda_k>0} \frac{\langle \Phi_n \mathbf{e}_n, \mathbf{u}_k\rangle \langle \mathbf{u}_k, \Phi_m \mathbf{e}_m\rangle}{\lambda - \lambda_k}, \tag{4.4}$$

where we have used that $\langle \mathbf{\Phi}'(x), \Phi_m \mathbf{e}_m\rangle = 0$. When there exists a zero eigenvalue of $\mathcal{A}(\lambda)$, there exists a solution $\mathbf{u} \in X_c^{(u)}(\mathbb{R})$, which is represented with the spectral decomposition in $L^2(\mathbb{R})$:

$$\mathbf{u}(x) = \sum_{m=1}^{N} \nu_m \left[\sum_{\lambda_{-k}<0} \frac{\langle \mathbf{u}_{-k}, \Phi_m \mathbf{e}_m\rangle}{\lambda - \lambda_{-k}} \mathbf{u}_{-k}(x) + \sum_{\lambda_k>0} \frac{\langle \mathbf{u}_k, \Phi_m \mathbf{e}_m\rangle}{\lambda - \lambda_k} \mathbf{u}_k(x) \right]. \tag{4.5}$$

Several lemmas follow from analysis of eigenvalues of $\mathcal{A}(\lambda)$.

Lemma 4.4 *The matrix eigenvalue problem $\mathcal{A}(\lambda)\boldsymbol{\nu} = \alpha(\lambda)\boldsymbol{\nu}$, $\lambda \in \mathbb{R}$ has N real eigenvalues $\alpha_1(\lambda)$, ..., $\alpha_N(\lambda)$, which are meromorphic functions of λ.*

Proof. The matrix $\mathcal{A}(\lambda)$ has N real eigenvalues $\alpha(\lambda)$ since it is Hermitian for $\lambda \in \mathbb{R}$ and $\lambda < 0$. Coefficients of $\mathcal{A}(\lambda)$ have pole singularities at $\lambda = \lambda_{-k}$ for $\lambda \leq 0$, unless $\langle \Phi_n \mathbf{e}_n, \mathbf{u}_{-k}\rangle = 0$, $n = 1, ..., N$. Since $\Phi_n \in L^2(\mathbb{R})$, $\mathbf{u} \in L^2(\mathbb{R})$, and $\langle \Phi_n \mathbf{e}_n, \mathbf{u}\rangle < \infty$, the series for $\mathcal{A}_{n,m}(\lambda)$ are bounded and converge for $\lambda \neq \lambda_{-k}$. In the limit $\lambda \to -\infty$, $\mathcal{A}_{n,m}(\lambda)$ converges to zero uniformly. As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ are meromorphic functions for $\lambda \leq 0$, which may have only pole singularities at $\lambda = \lambda_{-k}$. ∎

Lemma 4.5 *Eigenvalues $\alpha_1(\lambda)$, ..., $\alpha_N(\lambda)$ are decreasing functions of λ in the domain $\mathcal{D} = \{\lambda \leq 0 : \lambda \neq \lambda_{-k}, k = 1, ..., n(\mathcal{L}_1)\}$. All eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ are negative for $\lambda < \lambda_{-n(\mathcal{L}_1)}$.*

Proof. For Hermitian matrices, the set of eigenvalues $\{\alpha_n(\lambda)\}_{n=1}^N$ corresponds to the set of orthonormal eigenvectors $\{\boldsymbol{\nu}^{(n)}\}_{n=1}^N$ such that $\langle \boldsymbol{\nu}^{(n')}, \boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N} = \delta_{n',n}$, where $\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbb{C}^N} = \sum_{n=1}^N \bar{f}_n g_n$. We construct quadratic forms associated to the eigenvalue-eigenvector pairs $(\alpha_n, \boldsymbol{\nu}^{(n)})$, $n = 1, ..., N$:

$$\alpha_n(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}(\lambda)\boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N}, \quad \alpha_n'(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}'(\lambda)\boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N}. \tag{4.6}$$

Computing the derivative of $\mathcal{A}'(\lambda)$, we rewrite the second equality in (4.6) as

$$\alpha_n'(\lambda) = -\left(\sum_{\lambda_{-k}<0} \frac{b_{-k}}{(\lambda-\lambda_{-k})^2} + \sum_{\lambda_k>0} \frac{b_k}{(\lambda-\lambda_k)^2} \right) = -\langle \mathbf{u}^{(n)}, \mathbf{u}^{(n)} \rangle < 0, \tag{4.7}$$

where $\mathbf{u}^{(n)}(x)$ corresponds to $\boldsymbol{\nu} = \boldsymbol{\nu}^{(n)}$ and

$$b_{\pm k} = \left| \sum_{m=1}^{N} \langle \Phi_m \mathbf{e}_m, \mathbf{u}_{\pm k} \rangle \nu_m^{(n)} \right|^2 \geq 0. \tag{4.8}$$

As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ are decreasing functions of λ in the domain $\mathcal{D}$. In order to prove that all eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ are negative for $\lambda < \lambda_{-n(\mathcal{L}_1)}$, we find from (4.3) and (4.4) that

$$\lim_{\lambda \to -\infty} (\lambda \mathcal{A}_{n,m}(\lambda)) = \langle \Phi_n \mathbf{e}_n, \Phi_m \mathbf{e}_m \rangle = Q_{ns} \delta_{n,m}, \tag{4.9}$$

where $Q_{ns}(\boldsymbol{\beta}) = Q_n(\boldsymbol{\Phi})$ is defined by (1.5). It follows from the first equality in (4.6) and (4.9) that

$$\lim_{\lambda \to -\infty} \lambda \alpha_n(\lambda) = Q_{ns}(\boldsymbol{\beta}) > 0, \quad n = 1, ..., N,$$

such that $\lim_{\lambda \to -\infty} \alpha_n(\lambda) = -0$, $n = 1, ..., N$. Since eigenvalues $\alpha(\lambda)$ are continuous and decreasing for $\lambda < \lambda_{-n(\mathcal{L}_1)}$, they remain negative for all values of $\lambda < \lambda_{-n(\mathcal{L}_1)}$. ∎

Lemma 4.6 *Let λ_{-k} be a negative eigenvalue of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ with multiplicity q_{-k}, such that $q_{-k}^{\|}$ eigenvectors $\mathbf{u}_{-k}(x)$ belong to the constrained space $X_c^{(u)}(\mathbb{R})$ and $q_{-k}^{\perp} = (q_{-k} - q_{-k}^{\|})$ eigenvectors $\mathbf{u}_{-k}(x)$ belong to the orthogonal compliment of $X_c^{(u)}(\mathbb{R})$ in $L^2(\mathbb{R})$. There exist $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous at $\lambda = \lambda_{-k}$ and $q_{-k}^{\perp}$ eigenvalues $\alpha_n(\lambda)$ that have infinity discontinuities, jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$.*

Proof. In the limit $\lambda \to \lambda_{-k}$, we find that

$$\lim_{\lambda \to \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}_{n,m}(\lambda) = \sum_{r=1}^{q_{-k}} \langle \Phi_n \mathbf{e}_n, \mathbf{u}_{-k_r} \rangle \langle \mathbf{u}_{-k_r}, \Phi_m \mathbf{e}_m \rangle = \sum_{r=1}^{q_{-k}^{\perp}} \langle \Phi_n \mathbf{e}_n, \mathbf{u}_{-k_r} \rangle \langle \mathbf{u}_{-k_r}, \Phi_m \mathbf{e}_m \rangle. \tag{4.10}$$

Denote $\mathcal{B}_{-k} = \lim_{\lambda \to \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}(\lambda)$. The quadratic form $\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle_{\mathbb{C}^N}$ is diagonalized in normal variables,

$$x_r = \sum_{m=1}^{N} \langle \mathbf{u}_{-k_r}, \Phi_m \mathbf{e}_m \rangle \nu_m,$$

such that $\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle_{\mathbb{C}^N} = \sum_{r=1}^{q_{-k}^{\perp}} |x_r|^2$. Therefore, the matrix $\mathcal{B}_{-k}$ has exactly $q_{-k}^{\perp}$ positive eigenvalues and $(N - q_{-k}^{\perp})$ zero eigenvalues. Positive eigenvalues of $\mathcal{B}_{-k}$ correspond to $q_{-k}^{\perp}$ eigenvalues $\alpha_n(\lambda)$ jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$. Zero eigenvalues of $\mathcal{B}_{-k}$ correspond to $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous and have convergent Taylor series at $\lambda = \lambda_{-k}$. ∎

Lemma 4.7 *At $\lambda = 0$, there exist $p(\mathcal{U})$ positive, $z(\mathcal{U})$ zero, and $n(\mathcal{U})$ negative eigenvalues $\alpha_n(0)$, $n = 1, ..., N$.*

Proof. At $\lambda = 0$, the constrained eigenvalue problem (4.2) has an exact solution in $L^2(\mathbb{R})$:

$$\mathbf{u}_{\lambda=0}(x) = \sum_{m=1}^{N} \nu_m \frac{\partial \mathbf{\Phi}(x)}{\partial \beta_m} + c_0 \mathbf{\Phi}'(x), \tag{4.11}$$

where c_0 is not defined and

$$\mathcal{L}_1 \frac{\partial \mathbf{\Phi}}{\partial \beta_m} = -\Phi_m(x)\mathbf{e}_m. \tag{4.12}$$

Substituting (4.11) into (2.21), we find that $\mathcal{A}(0) = \frac{1}{2}\mathcal{U}$, where $\mathcal{U}$ is defined in (2.7). ■

Proof of Proposition 4.3. We consider eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ as meromorphic functions of λ for $\lambda \leq 0$. Starting with small negative values at $\lambda \to -\infty$, eigenvalues $\alpha_n(\lambda)$, $n = 1, ..., N$ decrease as λ increases toward $n(\mathcal{L}_1)$ pole singularities at $\lambda = \lambda_{-k}$. At each pole singularity, $q_{-k}^{\perp}$ eigenvalues $\alpha(\lambda)$ jump and pop up to the positive half-plane. The total number of jumping eigenvalues for $\lambda < 0$ is $\sum_{\lambda_{-k}<0} q_{-k}^{\perp}$. Only jumping eigenvalues may cross the value $\alpha(\lambda) = 0$, which corresponds to a negative eigenvalue λ of the constrained problem (4.2) in space $X_c^{(u)}(\mathbb{R})$. We find the total number of zeros of $\alpha(\lambda)$ at $\lambda \leq 0$ from Lemma 4.7 as

$$\sum_{\lambda_{-k}<0} q_{-k}^{\perp} - p(\mathcal{U}).$$

At each $\lambda = \lambda_{-k}$, there are $q_{-k}^{\parallel}$ eigenfunctions $\mathbf{u}_{-k}(x)$ that lie in the constrained space $X_c^{(u)}(\mathbb{R})$. Therefore, the total number of eigenvalues λ in $X_c^{(u)}(\mathbb{R})$ for $\lambda \leq 0$ is

$$\sum_{\lambda_{-k}} q_{-k}^{\perp} - p(\mathcal{U}) + \sum_{\lambda_{-k}} q_{-k}^{\parallel} = \sum_{\lambda_{-k}} q_{-k} - p(\mathcal{U}) = n(\mathcal{L}_1) - p(\mathcal{U}).$$

Subtracting the number $z(\mathcal{U})$ of zero eigenvalues at $\lambda = 0$, we prove the lemma. ■

Proposition 4.8 *Let $\mathcal{U}$ be the Hessian matrix with bounded eigenvalues. The kernel of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ lies in $X_c^{(u)}(\mathbb{R})$ for any $1 \leq z(\mathcal{L}_1) \leq N$.*

Proof. Suppose there exists an eigenvector $\mathbf{u}_0(x)$ of the kernel of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ such that $\mathbf{u}_0 \notin X_c^{(u)}(\mathbb{R})$. It follows from Lemma 4.6 that there exists an eigenvalue $\alpha_n(\lambda)$ that diverges at $\lambda \to -0$. It contradicts Lemma 4.7 since all eigenvalues of $\mathcal{U}$ are bounded. ■

Theorem 6 is proved with Propositions 4.2, 4.3, and 4.8. Using Theorem 6 and Proposition 4.1, we formulate the following proposition.

Proposition 4.9 *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_0$ or $\mathcal{L}_1$. There exists $\mathcal{L}$-invariant decomposition in $X_c^{(u)}(\mathbb{R})$, such that*

$$\forall \mathbf{u} \in X_c^{(u)}(\mathbb{R}) : \quad \mathbf{u}(x) = \sum_{m=1}^{\#_{<0}(\mathcal{L})} a_m \mathbf{u}_m(x) + \sum_{m=0}^{\#_{=0}(\mathcal{L})} b_m \mathbf{u}_m(x) + \mathbf{u}^+(x), \tag{4.13}$$

where $\mathbf{u}_m(x)$ are eigenvectors of $\mathcal{L}$ in $X_c^{(u)}(\mathbb{R})$ for negative and zero eigenvalues, and $\langle \mathbf{u}^+, \mathcal{L}\mathbf{u}^+ \rangle \geq c\langle \mathbf{u}^+, \mathbf{u}^+ \rangle$, $c > 0$. The quadratic form for $\mathcal{L}$ is diagonalized as follows,

$$\langle \mathbf{u}, \mathcal{L}\mathbf{u} \rangle = \sum_{m=1}^{\#_{<0}(\mathcal{L})} \lambda_m |a_m|^2 + \langle \mathbf{u}^+, \mathcal{L}\mathbf{u}^+ \rangle. \tag{4.14}$$

5 Eigenvalues of the linearized problem for $\mathcal{A}$

Here we study the spectrum of the non-self-adjoint linearized problem (2.14) with the simultaneous diagonalization of two self-adjoint operators $\mathcal{L}_1$ and $\mathcal{L}_0$. Similar studies were also considered in [G88, G90] but inertia law of matrix analysis was not exploited previously.

Lemma 5.1 *There exists a mapping $\gamma = -\lambda^2$ of the non-zero spectrum of $\mathcal{A}$ in $X_c(\mathbb{R})$ to the non-zero spectrum of the problem,*

$$\mathcal{L}_1 \mathbf{u} = \gamma \mathcal{L}_0^{-1} \mathbf{u}, \qquad \mathbf{u} \in X_c^{(u)}(\mathbb{R}). \tag{5.1}$$

Proof. Eigenvectors $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ form a basis in the kernel of $\mathcal{L}_0$. The operator $\mathcal{L}_0$ is invertible iff $\mathbf{u} \in X_c^{(u)}(\mathbb{R})$. It follows from (2.23) that $\mathbf{w} = \lambda \mathcal{L}_0^{-1}\mathbf{u}$, such that the problem (2.23) is equivalent to (5.1) for any $\lambda \neq 0$. Two eigenvectors $(\mathbf{u}, \mathbf{w})^T$ and $(\mathbf{u}, -\mathbf{w})^T$ of $\mathcal{A}$ in $X_c(\mathbb{R})$ corresponds to a single eigenvector $\mathbf{u}$ of the problem (5.1) in $X_c^{(u)}(\mathbb{R})$. ∎

Corollary 5.2 *Let $\gamma = \gamma_m$ be a non-zero eigenvalue of (5.1) with the eigenvector $\mathbf{u} = \mathbf{u}_m(x)$ in $X_c^{(u)}(\mathbb{R})$, such that*

$$\langle \mathbf{u}_m, \mathcal{L}_1 \mathbf{u}_m \rangle = \gamma_m \langle \mathbf{u}_m, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle. \tag{5.2}$$

Eigenvalue γ_m is real if either $\mathcal{L}_1$ or $\mathcal{L}_0$ is positive definite.

The problem (5.1) is a classical problem of simultaneous diagonalization of two self-adjoint operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$. Each operator can be orthogonally diagonalized due to Proposition 4.9. However, the orthogonal diagonalization (4.14) is relevant for the problem (5.1) only if the operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ commute, such that there exists a common basis for $\langle \mathbf{u}, \mathcal{L}_1 \mathbf{u} \rangle$ and $\langle \mathbf{u}, \mathcal{L}_0^{-1} \mathbf{u} \rangle$ [G61]. Since operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ do not commute, eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ in $X_c^{(u)}(\mathbb{R})$ do not generate eigenvectors of the problem (5.1). Moreover, complex eigenvalues and multiple eigenvalues with higher algebraic multiplicity may generally occur in the problem (5.1).

Lemma 5.3 *The spectrum of (5.1) is real if $\mathbf{\Phi}(x)$ is the ground state with $\Phi_n(x) > 0$ $\forall x \in \mathbb{R}$, $n = 1, ..., N$. Moreover, the positive-definite operator $\mathcal{L}_0$ can be factorized as $\mathcal{L}_0 = \mathcal{S}^+\mathcal{S}$, where $\mathcal{S}$ is a diagonal operator with the elements:*

$$\mathcal{S}_{n,m} = \sqrt{d_n}\left(\frac{\Phi_n'(x)}{\Phi_n(x)} - \frac{d}{dx}\right)\delta_{n,m}. \tag{5.3}$$

Proof. The factorization formula (5.3) follows from explicit computations:

$$\mathcal{S}_{n,n}^+\mathcal{S}_{n,n} = d_n\left(\frac{\Phi_n''}{\Phi_n} - \frac{d^2}{dx^2}\right) = (\mathcal{L}_0)_{n,n}.$$

Using the transformation $\mathbf{u} = \mathcal{S}^+\mathbf{v}$, we rewrite (5.1) in the form

$$\mathcal{S}\mathcal{L}_1\mathcal{S}^+\mathbf{v} = \gamma\mathbf{v}.$$

Since $\mathcal{S}\mathcal{L}_1\mathcal{S}^+$ is a self-adjoint operator, all eigenvalues γ are real. It is also clear from (5.3) that the kernel of $\mathcal{S}^+$ is empty, such that the transformation $\mathbf{u} = \mathcal{S}^+\mathbf{v}$ is invertible. ■

Lemma 5.4 *Let $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ be a complex eigenvalue of (5.1) such that $\gamma_{Rk}, \gamma_{Ik} \neq 0$, with a complex-valued eigenvector $\mathbf{u}_k(x) = \mathbf{u}_{Rk}(x) + i\mathbf{u}_{Ik}(x)$. A linear combination of two real-valued eigenvectors $\mathbf{u}(x) = a_k\mathbf{u}_{Rk}(x) + b_k\mathbf{u}_{Ik}(x)$ diagonalizes the quadratic forms $\langle \mathbf{u}, \mathcal{L}_1\mathbf{u}\rangle$ and $\langle \mathbf{u}, \mathcal{L}_0^{-1}\mathbf{u}\rangle$ with respect to Jordan blocks,*

$$\langle \mathbf{u}, \mathcal{L}_1\mathbf{u}\rangle = \mathbf{a}_k^T\hat{\gamma}_k\hat{l}_k\mathbf{a}_k, \qquad \langle \mathbf{u}, \mathcal{L}_0^{-1}\mathbf{u}\rangle = \mathbf{a}_k^T\hat{l}_k\mathbf{a}_k, \tag{5.4}$$

where $\mathbf{a}_k = (a_k, b_k)^T$,

$$\hat{\gamma}_k = \begin{pmatrix} \gamma_{Rk} & -\gamma_{Ik} \\ \gamma_{Ik} & \gamma_{Rk} \end{pmatrix}, \qquad \hat{l}_k = \begin{pmatrix} l_{Rk} & l_{Ik} \\ l_{Ik} & -l_{Rk} \end{pmatrix}, \tag{5.5}$$

and

$$l_{Rk} = \langle \mathbf{u}_{Rk}, \mathcal{L}_0^{-1}\mathbf{u}_{Rk}\rangle = -\langle \mathbf{u}_{Ik}, \mathcal{L}_0^{-1}\mathbf{u}_{Ik}\rangle, \tag{5.6}$$

$$l_{Ik} = \langle \mathbf{u}_{Ik}, \mathcal{L}_0^{-1}\mathbf{u}_{Rk}\rangle = \langle \mathbf{u}_{Rk}, \mathcal{L}_0^{-1}\mathbf{u}_{Ik}\rangle. \tag{5.7}$$

Proof. Since the quadratic forms $\langle \mathbf{u}_k, \mathcal{L}_1\mathbf{u}_k\rangle$ and $\langle \mathbf{u}_k, \mathcal{L}_0^{-1}\mathbf{u}_k\rangle$ in (5.2) are real-valued, the eigenvalues γ_k can be complex only if

$$\langle \mathbf{u}_k, \mathcal{L}_1\mathbf{u}_k\rangle = \langle \mathbf{u}_k, \mathcal{L}_0^{-1}\mathbf{u}_k\rangle = 0. \tag{5.8}$$

The zero inner product (5.8) for $\mathcal{L}_0^{-1}$ results in relations (5.6) and (5.7). The Jordan blocks (5.4)–(5.5) follow from direct computations. ■

Lemma 5.5 *Let $\gamma = \gamma_m$ be a real eigenvalue of (5.1) with a single real-valued eigenvector $\mathbf{u}_m(x)$ in $X_c^{(u)}(\mathbb{R})$. The eigenvalue $\gamma = \gamma_m$ is a multiple eigenvalue of higher algebraic multiplicity if and only if*

$$l_m = \langle \mathbf{u}_m, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = 0. \tag{5.9}$$

Proof. The eigenvalue $\gamma = \gamma_m$ is a degenerate eigenvalue of higher algebraic multiplicity if and only if there exists a solution of the derivative problem:

$$\mathcal{L}_1 \mathbf{u}'_m = \gamma_m \mathcal{L}_0^{-1} \mathbf{u}'_m + \mathcal{L}_0^{-1} \mathbf{u}_m, \qquad \mathbf{u}'_m \in X_c^{(u)}(\mathbb{R}). \tag{5.10}$$

Using the Fredholm Alternative Theorem with $\mathbf{u}_m(x)$, we arrive to the condition (5.9). ∎

A complex eigenvalue $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$, such that $\gamma_{Rk}, \gamma_{Ik} \neq 0$, with a single complex-valued eigenvector $\mathbf{u}_k(x) = \mathbf{u}_{Rk}(x) + i\mathbf{u}_{Ik}(x)$ is a multiple eigenvalue of higher algebraic multiplicity if and only if $l_{Rk} = l_{Ik} = 0$ in (5.6)–(5.7). According to Assumption 2.15(iii), we consider the generic case when non-zero eigenvalues of higher algebraic multiplicity do not occur in the problem (5.1).

Lemma 5.6 *Assume that $\sigma_{\text{dis}}(\mathcal{A})$ does not include non-zero eigenvalues of higher algebraic multiplicity. Eigenvectors $\mathbf{u}_m(x)$ for real eigenvalues γ_m and $(\mathbf{u}_{Rk}(x), \mathbf{u}_{Ik}(x))$ for complex eigenvalues $\gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ are orthogonal with respect to operator $\mathcal{L}_0^{-1}$,*

$$\langle \mathbf{u}_{m'}, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = l_m \delta_{m',m}, \tag{5.11}$$

and

$$\begin{aligned} \langle \mathbf{u}_{Rk'}, \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle = -\langle \mathbf{u}_{Ik'}, \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle &= l_{Rk} \delta_{k',k}, \\ \langle \mathbf{u}_{Ik'}, \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle = \langle \mathbf{u}_{Rk'}, \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle &= l_{Ik} \delta_{k',k}, \end{aligned} \tag{5.12}$$

where $l_m \neq 0$ and $|l_{Rk}|^2 + |l_{Ik}|^2 \neq 0$.

Proof. Orthogonality relations (5.11) and (5.12) for eigenvectors of the problem (5.1) follow from the identity:

$$(\gamma_{m'} - \gamma_m) \langle \mathbf{u}_{m'}, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = 0. \tag{5.13}$$

Coefficients l_m and $|l_{Rk}|^2 + |l_{Ik}|^2$ are non-zero due to Lemma 5.5, since γ_m and $\gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ are not eigenvalues of higher algebraic multiplicity. ∎

Corollary 5.7 *The set of eigenvectors $\mathbf{u}_m(x)$ and $(\mathbf{u}_{Rk}(x), \mathbf{u}_{Ik}(x))$ is also orthogonal with respect to operator $\mathcal{L}_1$.*

We shall also consider the quadratic forms $\langle \mathbf{u}, \mathcal{L}_1 \mathbf{u}\rangle$ and $\langle \mathbf{u}, \mathcal{L}_0^{-1}\mathbf{u}\rangle$ for eigenvectors of the continuous spectrum of the problem (5.1). Let us introduce the $\mathcal{A}$-invariant decomposition of $X_c^{(u)}(\mathbb{R})$ into the discrete part for $\sigma_{\rm p}(\mathcal{A})$ and the continuous part $Y_c^{(u)}(\mathcal{A})$ for $\sigma_{\rm con}(\mathcal{A})$:

$$X_c^{(u)}(\mathbb{R}) = \sum_{\lambda \in \sigma_{\rm p}(\mathcal{A})} \mathcal{N}_g(\mathcal{A} - \lambda) \oplus Y_c^{(u)}(\mathcal{A}), \qquad Y_c^{(u)}(\mathcal{A}) = \left[\sum_{\lambda \in \sigma_p(\mathcal{A})} \mathcal{N}_g(\mathcal{A}^* - \lambda) \right]^{\perp}, \tag{5.14}$$

where $\mathcal{A}^*$ is adjoint operator and $\sigma_{\rm p}(\mathcal{A}^*) = \sigma_{\rm p}(\mathcal{A})$. Using Assumption 2.15(i)–(ii), we have the following proposition.

Proposition 5.8 *Assume that the end points $\lambda = \pm i\beta_n$, $n = 1, ..., N$ are not resonances and $\sigma_{\rm ess}(\mathcal{A})$ does not include semi-eigenvalues nor embedded eigenvalues. The quadratic forms $\langle \mathbf{u}, \mathcal{L}_1 \mathbf{u}\rangle$ and $\langle \mathbf{u}, \mathcal{L}_0^{-1}\mathbf{u}\rangle$ are strictly positive in $Y_c^{(u)}(\mathcal{A})$, such that*

$$\forall \mathbf{u}^+ \in Y_c^{(u)}(\mathcal{A}) : \quad \langle \mathbf{u}^+, \mathcal{L}_1 \mathbf{u}^+ \rangle \geq c_1 \langle \mathbf{u}^+, \mathbf{u}^+\rangle, \qquad \langle \mathbf{u}^+, \mathcal{L}_0^{-1} \mathbf{u}^+ \rangle \geq c_0 \langle \mathbf{u}^+, \mathbf{u}^+\rangle, \tag{5.15}$$

where $c_1 > 0$, $c_0 > 0$.

Proof of Proposition 5.8 is given in Section 7 with the use of wave functions of the problem (2.14). Wave functions of $\mathcal{A}$ with $N = 1$ were introduced in [BP93], where orthogonality and completeness relations between the wave functions were derived by spectral analysis.

Combining Lemmas 5.4, 5.6, and Proposition 5.8, we formulate the following proposition.

Proposition 5.9 *Assume that Assumption 2.15 is satisfied. There exists $\mathcal{A}$-invariant decomposition in $X_c^{(u)}(\mathbb{R})$ such that*

$$\forall \mathbf{u} \in X_c^{(u)}(\mathbb{R}) : \quad \mathbf{u}(x) = \sum_k \left[a_k \mathbf{u}_{Rk}(x) + b_k \mathbf{u}_{Ik}(x) \right] + \sum_m c_m \mathbf{u}_m(x) + \mathbf{u}^+(x), \tag{5.16}$$

where $\mathbf{u}_m(x)$ are eigenvectors for real eigenvalues γ_m, $(\mathbf{u}_{Rk}(x), \mathbf{u}_{Ik}(x))$ are eigenvectors for complex eigenvalues $\gamma_k = \gamma_{Rk} + i\gamma_{Ik}$, and $\mathbf{u}^+ \in Y_c^{(u)}(\mathcal{A})$, such that (5.15) holds. The quadratic forms for $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ are simultaneously diagonalized as follows,

$$\langle \mathbf{u}, \mathcal{L}_1 \mathbf{u}\rangle = \sum_k \mathbf{a}_k^T \hat{\gamma}_k \hat{l}_k \mathbf{a}_k + \sum_m \gamma_m l_m |c_m|^2 + \langle \mathbf{u}^+, \mathcal{L}_1 \mathbf{u}^+\rangle, \tag{5.17}$$

$$\langle \mathbf{u}, \mathcal{L}_0^{-1} \mathbf{u}\rangle = \sum_k \mathbf{a}_k^T \hat{l}_k \mathbf{a}_k + \sum_m l_m |c_m|^2 + \langle \mathbf{u}^+, \mathcal{L}_0^{-1} \mathbf{u}^+\rangle, \tag{5.18}$$

where $\mathbf{a}_k = (a_k, b_k)^T$ and the Jordan blocks $\hat{\gamma}_k$ and $\hat{l}_k$ are defined by (5.5).

We note that Assumption 2.15(iii)–(v) excludes both non-zero and zero eigenvalues of higher algebraic multiplicity in the spectrum of $\mathcal{A}$, such that Jordan blocks for multiple eigenvalues are excluded from the decomposition (5.16). Jordan blocks for the zero eigenvalue of $\mathcal{A}$ with $N = 1$ were considered in [CP03].

6 Proof of Theorems 7 and 8

Eigenvalues γ of the diagonalization problem (5.1) correspond to three different types of eigenvalues λ of the linear stability problem (2.14). When $\gamma = \gamma_m > 0$, the linear problem (2.14) has two purely imaginary eigenvalues λ which are weakly spectrally stable. When $\gamma = \gamma_m < 0$, the linear problem (2.14) has two real eigenvalues λ, which include an unstable positive eigenvalue. When $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ is complex, the linear problem (2.14) has four complex eigenvalues λ, which include two unstable eigenvalues with $\mathrm{Re}(\lambda) > 0$. We trace the unstable eigenvalues λ of the stability problem (2.14) from negative and complex eigenvalues γ of the diagonalization problem (5.1), according to the following proposition.

Proposition 6.1 *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_1$ or $\mathcal{L}_0^{-1}$. The negative index $\#_{<0}(\mathcal{L})$ of the quadratic form $\langle \mathbf{u}, \mathcal{L}\mathbf{u} \rangle$ in Hilbert space $X_c^{(u)}(\mathbb{R})$ remains invariant in two diagonal representations (4.14) and (5.17)–(5.18).*

Proof. According to diagonalization (4.14), operator $\mathcal{L}$ has the basis $S_u = E_u^- \wedge E_0 \wedge E_u^+$, where E_u^- is the negative subspace spanned by eigenvectors $\{\mathbf{u}_m\}_{m=1}^{M_u}$ such that $\lambda_m = \langle \mathbf{u}_m, \mathcal{L}\mathbf{u}_m \rangle < 0$, E_0 is the kernel of $\mathcal{L}$ in $X_c^{(u)}(\mathbb{R})$, and E_u^+ is the positive subspace of $\mathcal{L}$ in $X_c^{(u)}(\mathbb{R})$. The negative index of $\langle \mathbf{u}, \mathcal{L}\mathbf{u} \rangle$ is $\#_{<0}(\mathcal{L}) = M_u$.

According to diagonalization (5.17) or (5.18), operator $\mathcal{L}$ has also another basis $S_v = E_v^c \wedge E_v^- \wedge E_0 \wedge E_v^+$, where E_v^- is the real negative subspace spanned by eigenvectors $\{\mathbf{v}_m\}_{m=1}^{M_v}$, such that $l_m = \langle \mathbf{v}_m, \mathcal{L}\mathbf{v}_m \rangle < 0$, E_v^c is the complex subspace spanned by eigenvectors $\{\mathbf{v}_{Rk}, \mathbf{v}_{Ik}\}_{k=1}^{K_v}$ such that $l_{Rk} = \langle \mathbf{v}_{Rk}, \mathcal{L}\mathbf{v}_{Rk} \rangle$, $l_{Ik} = \langle \mathbf{v}_{Ik}, \mathcal{L}\mathbf{v}_{Rk} \rangle$, and E_v^+ is the positive subspace of $\mathcal{L}$ in $X_c^{(u)}(\mathbb{R})$. The Jordan block $\hat{l}_k$ in (5.5) has one positive and one negative eigenvalues $\pm l_k = \pm\sqrt{l_{Rk}^2 + l_{Ik}^2}$, such that the negative index of $\langle \mathbf{u}, \mathcal{L}\mathbf{u} \rangle$ is $\#_{<0}(S_v) = M_v + K_v$. The eigenvectors $\mathbf{v}_{Rk}(x)$ and $\mathbf{v}_{Ik}(x)$ can be orthogonalized with respect to operator $\mathcal{L}$ in the linear combination:

$$\mathbf{v}_k^{\pm} = l_{Ik}\mathbf{v}_{Rk}(x) + (\pm l_k - l_{Rk})\,\mathbf{v}_{Ik}(x).$$

We assume that $(M_v + K_v) > M_u$ and show that this is false. The case $(M_v + K_v) < M_u$ can be treated similarly. Consider a function $\mathbf{g}_u(x)$ given by

$$\mathbf{g}_u(x) = \sum_{k=1}^{K_v} a_k \mathbf{v}_k^-(x) + \sum_{m=1}^{M_v} c_m \mathbf{v}_m(x) + \mathbf{u}_0 + \mathbf{u}^+(x). \tag{6.1}$$

The eigenfunctions $\mathbf{v}_k^-(x)$ and $\mathbf{v}_m(x)$ can be decomposed over the basis of S_u:

$$\mathbf{v}_k^-(x) = \sum_{j=1}^{M_u} \alpha_{kj}\mathbf{u}_j(x) + \mathbf{u}_{0k}(x) + \mathbf{u}_k^+(x), \tag{6.2}$$

$$\mathbf{v}_m(x) = \sum_{j=1}^{M_u} \gamma_{mj}\mathbf{u}_j(x) + \mathbf{u}_{0m}(x) + \mathbf{u}_m^+(x). \tag{6.3}$$

Therefore, the function $\mathbf{g}_u(x)$ is decomposed as:

$$\begin{aligned}
\mathbf{g}_u(x) &= \sum_{j=1}^{M_u} \left(\sum_{k=1}^{K_v} \alpha_{kj} a_k + \sum_{m=1}^{M_v} \gamma_{mj} c_m \right) \mathbf{u}_j(x) \\
&+ \left(\mathbf{u}_0(x) + \sum_{k=1}^{K_v} a_k \mathbf{u}_{0k}(x) + \sum_{m=1}^{M_v} c_m \mathbf{u}_{0m}(x) \right) \\
&+ \left(\mathbf{u}^+(x) + \sum_{k=1}^{K_v} a_k \mathbf{u}_k^+(x) + \sum_{m=1}^{M_v} c_m \mathbf{u}_m^+(x) \right). \qquad (6.4)
\end{aligned}$$

Consider a particular case $\mathbf{g}_u(x) = \mathbf{0}$. Since the set S_u is complete, then

$$\begin{aligned}
\sum_{k=1}^{K_v} \alpha_{kj} a_k + \sum_{m=1}^{M_v} \gamma_{mj} c_m &= 0, \quad j = 1, ..., M_u, \qquad (6.5) \\
\mathbf{u}_0(x) + \sum_{k=1}^{K_v} a_k \mathbf{u}_{0k}(x) + \sum_{m=1}^{M_v} c_m \mathbf{u}_{0m}(x) &= 0, \\
\mathbf{u}_+(x) + \sum_{k=1}^{K_v} a_k \mathbf{u}_k^+(x) + \sum_{m=1}^{M_v} c_m \mathbf{u}_m^+(x) &= 0.
\end{aligned}$$

The linear homogeneous system (6.5) is under-determined, such that at least $(M_v + K_v - M_u)$ unknowns are arbitrary. Therefore, there exists a non-zero solution of (6.5) such that a non-zero vector $\mathbf{s}_u(x)$ is defined by (6.1) with $\mathbf{g}_u(x) = \mathbf{0}$:

$$\mathbf{s}_u(x) = \sum_{k=1}^{K_v} a_k \mathbf{v}_k^-(x) + \sum_{m=1}^{M_v} c_m \mathbf{v}_m(x) = -\mathbf{u}_0(x) - \mathbf{u}^+(x).$$

Therefore, the quadratic form $\langle \mathbf{s}_u, \mathcal{L}\mathbf{s}_u \rangle$ can be bounded by two contradictory ways:

$$\begin{aligned}
\langle \mathbf{s}_u, \mathcal{L}\mathbf{s}_u \rangle &= -\sum_{k=1}^{K_v} 2 \left(l_{Rk}^2 + l_{Ik}^2 \right) \left(\sqrt{l_{Rk}^2 + l_{Ik}^2} + l_{Rk} \right) |a_k|^2 + \sum_{m=1}^{M_v} l_m |c_m|^2 < 0, \\
\langle \mathbf{s}_u, \mathcal{L}\mathbf{s}_u \rangle &= \langle \mathbf{u}^+, \mathcal{L}\mathbf{u}^+ \rangle > 0.
\end{aligned}$$

The contradiction is resolved if and only if $(M_v + K_v) = M_u$. ■

Corollary 6.2 *Let $N_{\rm comp}$ be the number of complex eigenvalues in the problem (5.1). Let $\#_{<0}(\mathcal{L}_0)$ and $\#_{<0}(\mathcal{L}_1)$ be the numbers of negative eigenvalues in the problems (4.1) and (4.2). Then $N_{\rm comp} \leq \min(\#_{<0}(\mathcal{L}_0), \#_{<0}(\mathcal{L}_1))$ and there exists $(\#_{<0}(\mathcal{L}_1) - N_{\rm comp})$ eigenvectors $\mathbf{u}_m(x)$ in the problem (5.1) such that $\langle \mathbf{u}_m, \mathcal{L}_1 \mathbf{u}_m \rangle < 0$ and $(\#_{<0}(\mathcal{L}_0) - N_{\rm comp})$ eigenvectors $\mathbf{u}_m(x)$ such that $\langle \mathbf{u}_m, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle < 0$.*

Proposition 6.1 generalizes the Sylvester's Inertia Theorem for finite-dimensional operators [G61]. Using this result, we prove Theorems 7 and 8, which define sharp bounds on the number of negative and complex eigenvalues of the problem (5.1) from the numbers of negative eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_0$.

Proof of Theorem 7. It follows from Corollary 6.2 that

$$\#_{<0}(\mathcal{L}_1) = N_{\rm comp} + \#_{<0}(\gamma_m l_m), \tag{6.6}$$
$$\#_{<0}(\mathcal{L}_0) = N_{\rm comp} + \#_{<0}(l_m). \tag{6.7}$$

Taking the sum of (6.6) and (6.7), we find that

$$\#_{<0}(\mathcal{L}_1) + \#_{<0}(\mathcal{L}_0) = 2N_{\rm comp} + \#_{<0}(\gamma_m l_m) + \#_{<0}(l_m) = 2N_{\rm comp} + 2N_{\rm imag}^- + N_{\rm real}.$$

By Theorem 6, the latter identity gives the closure relation (3.1). ■

Proof of Theorem 8. Taking the difference of (6.6) and (6.7), we find that

$$|\#_{<0}(\mathcal{L}_1) - \#_{<0}(\mathcal{L}_0)| = |\#_{<0}(\gamma_m l_m) - \#_{<0}(l_m)| \leq N_{\rm real} \leq N_{\rm unst},$$

which is the lower bounds (3.2) and (3.3). The upper bound in (3.2) is a corollary of Theorem 7. The bound (3.4) is given by Corollary 6.2. ■

We note that the constrained problems (4.1) and (4.2) have a common set of eigenfunctions if and only if operators $\mathcal{L}_1$ and $\mathcal{L}_0$ commute. Assume that this is true. Let $\lambda = \xi_m$ be an eigenvalue of (4.1) and $\lambda = \eta_m$ be an eigenvalue of (4.2), with the same eigenvector $\mathbf{u}_m(x)$. There exists an eigenvalue $\gamma = \gamma_m$ of the problem (5.1), such that

$$\gamma_m = \eta_m \xi_m = \eta_m \frac{\langle \mathbf{u}_m, \mathcal{L}_0 \mathbf{u}_m \rangle}{\langle \mathbf{u}_m, \mathbf{u}_m \rangle} = \eta_m \frac{\langle \mathbf{u}_m, \mathbf{u}_m \rangle}{\langle \mathbf{u}_m, \mathcal{L}_0^{-1} \mathbf{u}_m \rangle}. \tag{6.8}$$

This formula was used in [PK00] to approximate $\lambda^2 = -\gamma_m$ from the given solution of the constrained problem (4.2).

7 Proof of Proposition 5.8.

We introduce wave functions of the spectral problem (2.14), similarly to analysis in [BP93] for $N = 1$. Since operators $\mathcal{L}_0$ and $\mathcal{L}_1$ in (2.12) and (2.13) are diagonal differential operators with exponentially decaying matrix potentials, there exist $2N$ branches of the continuous spectrum, located symmetrically at

$$\sigma_{\rm con}(\mathcal{A}) = \cup_{1 \leq n \leq N} \{\lambda \in i\mathbb{R} : \; |{\rm Im}(\lambda)| \geq \beta_n\}. \tag{7.1}$$

Using a transformation: $\lambda \mapsto i\Omega$, $\mathbf{u} \mapsto \mathbf{u}$, $\mathbf{w} \mapsto i\mathbf{w}$, we rewrite the problem (2.14) as

$$\mathcal{L}_1 \mathbf{u} = \Omega \mathbf{w}, \qquad \mathcal{L}_0 \mathbf{w} = \Omega \mathbf{u}. \tag{7.2}$$

We use the order

$$\beta_1 \leq \beta_2 \leq ... \leq \beta_N \tag{7.3}$$

and denote the number of end points β_n to the left of the given value Ω by K_Ω, $K_\Omega \leq N$. Let $\Omega_E = \{\beta_1, ..., \beta_N\}$. We introduce a set of continuous parameters $k_n \in \mathbb{R}$, $n = 1, ..., N$, where k_n parameterizes the n-th positive branch of the continuous spectrum:

$$\Omega = \beta_n + d_n k_n^2, \qquad n = 1, ..., N, \tag{7.4}$$

such that

$$k_n = k_n(\Omega) = \left(\frac{\Omega - \beta_n}{d_n}\right)^{1/2}. \tag{7.5}$$

It is clear that two n-th positive branches exist for each $\Omega > \beta_n$, with $k_n > 0$ and $k_n < 0$. We consider the branch $k_n > 0$ and define a set of wave functions $\mathbf{u}_n^{\pm}(\Omega) \equiv \mathbf{u}_n^{\pm}(x; \mathbf{k}(\Omega))$ for $\Omega \neq \Omega_E$, according to asymptotic values at infinity:

$$\mathbf{u}_n^{\pm}(\Omega) \to \mathbf{e}_n e^{\pm i k_n x} \quad \text{as} \quad x \to \pm\infty, \qquad k_n > 0, \tag{7.6}$$

where $\mathbf{e}_n$ is the n-th unit vector in $\mathbb{C}^N$. We define a set of scattering coefficients from asymptotic values of $\mathbf{u}_n^{\pm}(k)$ at the other infinities:

$$\begin{aligned} \mathbf{u}_n^{-}(\Omega) &\to \sum_{l=1}^{K_\Omega} \mathbf{e}_l \left[a_{n,l}(\Omega) e^{-i k_l x} + b_{n,l}(\Omega) e^{i k_l x} \right] \quad \text{as} \quad x \to +\infty, \\ \mathbf{u}_n^{+}(\Omega) &\to \sum_{l=1}^{K_\Omega} \mathbf{e}_l \left[A_{n,l}(\Omega) e^{i k_l x} + B_{n,l}(\Omega) e^{-i k_l x} \right] \quad \text{as} \quad x \to -\infty, \end{aligned} \tag{7.7}$$

where $k_l > 0$, $l = 1, ..., K_\Omega$, $\Omega \neq \Omega_E$. It follows from the system (7.2) that the components $\mathbf{w}_n^{\pm}(\Omega)$ have the same asymptotic representation (7.6)–(7.7) for the branch of the continuous spectrum (7.4) with $\Omega > \beta_n$.

Definition 7.1 *When the eigenvector of (7.2) with $\Omega \geq \beta_1$ is exponentially decaying, Ω is called an embedded eigenvalue of $\sigma_{\rm emb}(\mathcal{A})$. When the eigenvector of (7.2) with $\Omega \geq \beta_1$ is exponentially decaying at one infinity and bounded at the other infinity, Ω is called a semi-eigenvalue of $\sigma_{\rm con}(\mathcal{A})$. When the set of wave functions $\{\mathbf{u}_n^{-}(\Omega)\}_{n=1}^{K_\Omega}$, $K_\Omega \leq N$ is linearly dependent from the set of wave functions $\{\mathbf{u}_n^{+}(\Omega)\}_{n=1}^{K_\Omega}$, Ω is called a resonance of $\sigma_{\rm con}(\mathcal{A})$.*

According to Assumption 2.15(i)–(ii), we assume that the end points $\Omega = \Omega_E$ are not resonances and semi-eigenvalues or embedded eigenvalues do not exist for $\Omega \geq \beta_1$.

Existence of wave functions was shown in [BP93] for $N = 1$, where all fundamental solutions of the linear system (7.2) were considered with Volterra integral equations, including exponentially decreasing and increasing solutions. Exponentially decreasing terms are neglected in the asymptotic representations (7.6)–(7.7). Existence of the wave functions $\mathbf{u}_{\pm}^{(m)}(\Omega)$ is addressed by the following lemma.

Lemma 7.2 *Assume that no semi-eigenvalues and embedded eigenvalues exist for $\Omega \geq \beta_1$. The wave functions $\mathbf{u}_n^{\pm}(\Omega)$, $n = 1, ..., N$ exist and have unique asymptotic representations (7.6)–(7.7), where coefficients $a_{n,l}(\Omega)$, $b_{n,l}(\Omega)$ are bounded for any $\Omega > \beta_n$, $\Omega \neq \Omega_E$.*

Proof. For $\Omega > \beta_n$, besides a set of $2K_\Omega$ oscillatory functions, $K_\Omega \leq N$, there exist sets of $2N - K_\Omega$ exponentially decaying and $2N - K_\Omega$ exponentially growing solutions at each infinity $x \to \pm\infty$. When $\Omega \neq \Omega_E$, all functions are uniquely defined by standard theorems on solutions of linear differential equations with exponentially decaying coefficients [CL55]. In order to define $\mathbf{u}_n^{\pm}(\Omega)$, we construct a linear combination of $2N - K_\Omega$ exponentially decaying functions at $x \to \pm\infty$ with the oscillatory function $\mathbf{e}_n e^{\pm i k_n x}$ and uniquely define the coefficients of the linear combination from $2N - K_\Omega$ conditions that exponentially growing functions are removed in the limit $x \to \mp\infty$. If semi-eigenvalues and embedded eigenvalues do not exist for $\Omega > \beta_n$, the non-homogeneous linear system for the coefficients always has a unique solution. Therefore, the wave functions $\mathbf{u}_n^{\pm}(\Omega)$ are uniquely specified by the asymptotic representations (7.6)–(7.7) for any $\Omega > \beta_n$, $\Omega \neq \Omega_E$ and the coefficients $a_{n,l}(\Omega)$, $b_{n,l}(\Omega)$, $l = 1, ..., K_\Omega$ are bounded. ∎

We define a "scalar" Wronskian between two solutions $\mathbf{u}^{(1)}(x)$, $\mathbf{u}^{(2)}(x)$ of the system (7.2) with Ω_1, Ω_2:

$$W[\mathbf{u}^{(1)}, \mathbf{u}^{(2)}] = \sum_{n=1}^{N} d_n \left(u_n^{(1)} u_n^{(2)\prime} - u_n^{(1)\prime} u_n^{(2)} + w_n^{(1)} w_n^{(2)\prime} - w_n^{(1)\prime} w_n^{(2)} \right). \tag{7.8}$$

It follows from the system (7.2) that

$$\frac{d}{dx} W[\mathbf{u}^{(1)}, \mathbf{u}^{(2)}] = (\Omega_1 - \Omega_2) \sum_{n=1}^{N} \left(u_n^{(1)} w_n^{(2)} + w_n^{(1)} u_n^{(2)} \right). \tag{7.9}$$

If $\Omega_1 = \Omega_2$, then $W[\mathbf{u}^{(1)}, \mathbf{u}^{(2)}]$ is constant in $x \in \mathbb{R}$. Using asymptotic values (7.6),(7.7) for $W[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^+(\Omega)]$ and $W[\bar{\mathbf{u}}_m^-(\Omega), \mathbf{u}_n^+(\Omega)]$, we derive the linear relations between the scattering coefficients:

$$A_{n,m}(\Omega) = \frac{k_n d_n}{k_m d_m} a_{m,n}(\Omega), \qquad B_{n,m}(\Omega) = -\frac{k_n d_n}{k_m d_m} \bar{b}_{m,n}(\Omega). \tag{7.10}$$

Using asymptotic values (7.6)–(7.7) for $W[\bar{\mathbf{u}}_m^-(\Omega), \mathbf{u}_n^-(\Omega)]$ and $W[\bar{\mathbf{u}}_m^+(\Omega), \mathbf{u}_n^+(\Omega)]$, we derive the quadratic relations between the scattering coefficients:

$$k_n d_n \delta_{m,n} = \sum_{l=1}^{K_\Omega} k_l d_l \left[\bar{a}_{m,l}(\Omega) a_{n,l}(\Omega) - \bar{b}_{m,l}(\Omega) b_{n,l}(\Omega) \right] \tag{7.11}$$

and

$$\frac{1}{k_n d_n} \delta_{m,n} = \sum_{l=1}^{K_\Omega} \frac{1}{k_l d_l} \left[\bar{a}_{l,m}(\Omega) a_{l,n}(\Omega) - \bar{b}_{l,m}(\Omega) b_{l,n}(\Omega) \right]. \tag{7.12}$$

We use the scalar Wronskians (7.8) to study the behavior of wave functions at the end points and to derive the orthogonality relations between the wave functions.

Lemma 7.3 *No resonances may occur for any $\Omega \geq \beta_1$, $\Omega \neq \Omega_E$.*

Proof. According to Definition 7.1, Ω is a resonance, if there exists a non-zero eigenvector $\mathbf{u}(x)$ such that

$$\mathbf{u}(x) = \sum_{n=1}^{K_\Omega} c_n^- \mathbf{u}_n^-(\Omega) = \sum_{n=1}^{K_\Omega} c_n^+ \mathbf{u}_n^+(\Omega). \tag{7.13}$$

Computing $W[\bar{\mathbf{u}}, \mathbf{u}]$ in the limits $x \to \pm\infty$, we have

$$\sum_{n=1}^{K_\Omega} k_n \left(|c_n^-|^2 + |c_n^+|^2\right) = 0. \tag{7.14}$$

When $\Omega \neq \Omega_E$, all $k_n > 0$, such that all $c_n^\pm = 0$. Therefore, no eigenvector $\mathbf{u}(x)$ exists for $\Omega \neq \Omega_E$. ∎

Lemma 7.4 *When the end point $\Omega = \beta_n$, $n = 1, ..., N$ is not a resonance, the scattering coefficients $a_{n,n}(\Omega)$, $b_{n,n}(\Omega)$ diverges in the limit $\Omega \to \beta_n$ as follows:*

$$\lim_{\Omega \to \beta_n + 0} k_n a_{n,n}(\Omega) = - \lim_{\Omega \to \beta_n + 0} k_n b_{n,n}(\Omega) = \hat{\alpha}_n, \tag{7.15}$$

where $\hat{\alpha}_n \neq 0$.

Proof. Assume for convenience that $\beta_{n-1} < \beta_n < \beta_{n+1}$. At $\Omega = \beta_n$, such that $k_n = 0$, we have a modified asymptotic representation for $\mathbf{u}_m^-(\Omega)$ as $x \to +\infty$:

$$\mathbf{u}_m^-(\Omega) \to \sum_{l=1}^{K_\Omega} \mathbf{e}_l \left[a_{m,l}(\Omega) e^{-ik_l x} + b_{m,l}(\Omega) e^{ik_l x}\right] + \mathbf{e}_n \left[\hat{b}_{m,n} + \hat{a}_{m,n} x\right], \qquad m = 1, ..., n, \tag{7.16}$$

where $n = K_\Omega + 1$, and

$$\begin{aligned} \hat{b}_{m,n} &= \lim_{\Omega \to \beta_n + 0} \left(b_{m,n}(\Omega) + a_{m,n}(\Omega)\right), \\ \hat{a}_{m,n} &= \lim_{\Omega \to \beta_n + 0} i k_n \left(b_{m,n}(\Omega) - a_{m,n}(\Omega)\right). \end{aligned} \tag{7.17}$$

For $\Omega = \beta_n - 0$, no resonance may occur, such that the coefficient matrix $[a_{m,l}]_{1 \leq m,l \leq K_\Omega}$ is not singular. By continuity, it remains non-singular for $\Omega = \beta_n + 0$. Since $\Omega = \beta_n$ is not a resonance, the extended coefficient matrix for $1 \leq m, l \leq n$ is also non-singular, and therefore $\hat{a}_{n,n} = -2i\hat{\alpha}_n \neq 0$. ∎

We define the symplectic inner product as

$$J[\mathbf{u}^{(1)}, \mathbf{u}^{(2)}] = \frac{1}{2\pi i} \langle \mathbf{u}^{(1)}, \mathcal{J} \mathbf{u}^{(2)} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{N} \left(\bar{u}_n^{(1)} w_n^{(2)} + \bar{w}_n^{(1)} u_n^{(2)}\right) dx. \tag{7.18}$$

The Dirac-function $\delta(k)$ has the properties:

$$\delta(k) = \frac{1}{\pi} \lim_{L \to \infty} \frac{e^{ikL}}{ik} \tag{7.19}$$

and $|\alpha| \delta(\alpha k) = \delta(k)$, $\alpha \neq 0$. Using standard computations of the symplectic inner products (7.18), we derive the orthogonality relations between wave functions $\mathbf{u}_n^\pm(\Omega)$, $n = 1, ..., N$.

Lemma 7.5 *Let the set of wave functions* $\{\mathbf{u}_n^{\pm}(\Omega)\}_{n=1}^N$ *be defined by (7.6)-(7.7) for* $k_n > 0$. *The following orthogonality relations hold:*

$$\begin{aligned} J[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^-(\Omega')] &= \alpha_{m,n}(\Omega)\delta(\Omega - \Omega'), & (7.20)\\ J[\mathbf{u}_m^+(\Omega), \mathbf{u}_n^+(\Omega')] &= \beta_{m,n}(\Omega)\delta(\Omega - \Omega'), & (7.21)\\ J[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^+(\Omega')] &= 0. & (7.22) \end{aligned}$$

where

$$\alpha_{m,n}(\Omega) = 4\sum_{l=1}^{K_\Omega} k_l d_l \bar{a}_{m,l}(\Omega) a_{n,l}(\Omega), \qquad \beta_{m,n}(\Omega) = 4\sum_{l=1}^{K_\Omega} \frac{1}{k_l d_l} \bar{a}_{l,m}(\Omega) a_{l,n}(\Omega). \tag{7.23}$$

Proof. We integrate the Wronskian relation (7.9) as

$$J[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^-(\Omega')] = \frac{1}{2\pi}\lim_{x\to\infty} \frac{W[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^-(\Omega')]}{\Omega - \Omega'} - \frac{1}{2\pi}\lim_{x\to-\infty} \frac{W[\mathbf{u}_m^-(\Omega), \mathbf{u}_n^-(\Omega')]}{\Omega - \Omega'}. \tag{7.24}$$

The second term in (7.24) is computed with the use of (7.4), (7.6) and (7.19) as $\delta_{m,n}\delta(k_n - k_n')$. The first term in (7.24) is computed with the use of (7.4), (7.7) and (7.19) as

$$\sum_{l=1}^{K_\Omega} \left(\bar{a}_{m,l} a_{n,l} + \bar{b}_{m,l} b_{n,l}\right) \delta(k_l - k_l') + \sum_{l=1}^{K_\Omega} \left(\bar{a}_{m,l} b_{n,l} + \bar{b}_{m,l} a_{n,l}\right) \delta(k_l + k_l'),$$

where we suppressed the arguments of $a_{n,l}(\Omega)$ and $b_{n,l}(\Omega)$. Since $k_l > 0$ and $k_l' > 0$ in the representation (7.7), we understand that $\delta(k_l + k_l') = 0$ and

$$\delta(k_l - k_l') = 2k_l d_l \delta(\Omega - \Omega').$$

Using (7.11), we derive (7.20). The other relations (7.21) and (7.22) are derived similarly, with the use of (7.10) and (7.12). ∎

Lemma 7.6 *The coefficient matrices* $[\alpha_{m,n}(\Omega)]_{1\le m,n\le K_\Omega}$ *and* $[\beta_{m,n}(\Omega)]_{1\le m,n\le K_\Omega}$ *are strictly positive for* $\Omega \neq \Omega_E$.

Proof. We consider a quadratic form in $\mathbb{C}^{K_\Omega}$:

$$\sum_{m=1}^{K_\Omega}\sum_{n=1}^{K_\Omega} \bar{x}_m \alpha_{n,m}(\Omega) x_n = 4\sum_{l=1}^{K_\Omega} k_l d_l \left| \sum_{n=1}^{K_\Omega} a_{n,l}(\Omega) x_n \right|^2 \ge 0. \tag{7.25}$$

Since all $k_l > 0$, $l = 1, ..., K_\Omega$ for $\Omega \neq \Omega_E$, the equality would mean that the determinant of the matrix $[a_{m,n}(\Omega)]_{1\le m,n\le K_\Omega}$ is zero, which contradicts to Lemma 7.3. Therefore, the quadratic form in (7.25) is strictly positive for $\Omega \neq \Omega_E$. Similar computations hold for the matrix $[\beta_{m,n}(\Omega)]_{1\le m,n\le K_\Omega}$. ∎

We define the normalized set of wave functions $\mathbf{e}_n(\Omega)$, such that

$$\begin{aligned}
\mathbf{e}_n(\Omega) \equiv \mathbf{e}_n^+(\Omega) &= \frac{1}{\sqrt{k_n d_n \beta_{n,n}}} \mathbf{u}_n^+(\Omega), \quad k_n \geq 0, \\
\mathbf{e}_n(\Omega) \equiv \mathbf{e}_n^-(\Omega) &= \sqrt{\frac{k_n d_n}{\alpha_{n,n}}} \mathbf{u}_n^-(\Omega), \quad k_n \leq 0.
\end{aligned}$$

It follows from (7.11)–(7.12) and (7.23) at $m = n$ that $\alpha_{n,n}(\Omega) > 0$ and $\beta_{n,n}(\Omega) > 0$. Lemma 7.4 implies that $\mathbf{e}_n(\Omega) = \mathrm{O}(k_n)$ as $k_n \to 0$, such that the normalized wave functions $\mathbf{e}_n(\Omega)$ are continuous at $k_n = 0$. It follows from Lemma 7.5 that the wave functions $\{\mathbf{e}_n(\Omega)\}_{n=1}^N$ satisfy the orthogonality relations:

$$J[\mathbf{e}_m(\Omega), \mathbf{e}_n(\Omega')] = \rho_{m,n}(\Omega)\delta(\Omega - \Omega'), \qquad k_n \in \mathbb{R}, \tag{7.26}$$

where

$$\begin{aligned}
\rho_{m,n}(\Omega) \equiv \rho_{m,n}^+(\Omega) &= \frac{\beta_{m,n}}{\sqrt{k_m k_n d_m d_n \beta_{m,m} \beta_{n,n}}}, \qquad k_n \geq 0, \\
\rho_{m,n}(\Omega) \equiv \rho_{m,n}^-(\Omega) &= \sqrt{\frac{k_m k_n d_m d_n}{\alpha_{m,m} \alpha_{n,n}}} \alpha_{m,n}, \qquad k_n \leq 0.
\end{aligned} \tag{7.27}$$

It is clear from Lemma 7.6 that the matrix $[\rho_{m,n}(\Omega)]_{1 \leq m,n \leq K_\Omega}$ is also positive for any $\Omega \in \sigma_{\mathrm{con}}(\mathcal{A})$. We define the projection operator $\mathcal{S} : X_c^{(u)}(\mathbb{R}) \mapsto Y_c^{(u)}(\mathcal{A})$, according to standard formula:

$$\forall \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \exists \mathbf{u}^+ \in Y_c^{(u)}(\mathcal{A}) : \quad \mathbf{u}^+ = \mathcal{S}\mathbf{u} = \sum_{n=1}^N \int_{-\infty}^{\infty} \hat{u}_n(\Omega) \mathbf{e}_n(\Omega) dk_n, \tag{7.28}$$

where coefficients $\hat{u}_n(\Omega)$ are uniquely defined by the orthogonality relations (7.26), since the matrix $[\rho_{m,n}(\Omega)]_{1 \leq m,n \leq K_\Omega}$ is positive definite for any $\Omega \in \sigma_{\mathrm{con}}(\mathcal{A})$. Using (7.4), we can rewrite (7.28) in the form:

$$\mathbf{u}^+(x) = \sum_{n=1}^N \int_{\beta_n}^{\infty} \frac{d\Omega}{2k_n d_n} \left(\hat{u}_n^+(\Omega) \mathbf{e}_n^+(\Omega) + \hat{u}_n^-(\Omega) \mathbf{e}_n^-(\Omega) \right), \qquad k_n > 0. \tag{7.29}$$

We note that the integrals (7.29) are not singular at the end points $\Omega = \beta_n$, since $\mathbf{e}_n^{\pm}(\Omega) = \mathrm{O}(k_n)$ as $k_n \to 0$. With this construction, we finally prove Proposition 5.8.

Proofof Proposition 5.8. Using (7.2) and (7.18), we find that

$$\rho_{m,n}(\Omega)\delta(\Omega - \Omega') = J[\mathbf{e}_m(\Omega), \mathbf{e}_n(\Omega')] = \frac{\langle \mathbf{e}_m(\Omega), \mathcal{L}_1 \mathbf{e}_n(\Omega') \rangle}{2\pi\Omega'} + \frac{\langle \mathcal{L}_1 \mathbf{e}_m(\Omega), \mathbf{e}_n(\Omega') \rangle}{2\pi\Omega}. \tag{7.30}$$

Integrating by parts and using quadratic relations (7.11)–(7.12) for asymptotic representations (7.6)–(7.7), we confirm that

$$\langle \mathcal{L}_1 \mathbf{e}_m(\Omega), \mathbf{e}_n(\Omega') \rangle = \langle \mathbf{e}_m(\Omega), \mathcal{L}_1 \mathbf{e}_n(\Omega') \rangle. \tag{7.31}$$

As a result, we have simultaneous orthogonality relations:

$$\begin{aligned}
\langle \mathbf{e}_m(\Omega), \mathcal{L}_1 \mathbf{e}_n(\Omega') \rangle &= \pi \Omega \rho_{m,n}(\Omega) \delta(\Omega - \Omega'), \\
\langle \mathbf{e}_m(\Omega), \mathcal{L}_0^{-1} \mathbf{e}_n(\Omega') \rangle &= \frac{\pi}{\Omega} \rho_{m,n}(\Omega) \delta(\Omega - \Omega').
\end{aligned} \tag{7.32}$$

A simple calculation of quadratic form $\langle \mathbf{u}^+, \mathcal{L}_1 \mathbf{u}^+ \rangle$ for $\mathbf{u}^+ \in Y_c^{(u)}(\mathcal{A})$ with the use of the spectral representation (7.29) and the orthogonality relations (7.32) leads to formula:

$$\langle \mathbf{u}^+, \mathcal{L}_1 \mathbf{u}^+ \rangle = \int_{\beta_1}^{\infty} \left(\sum_{m=1}^{K_\Omega} \sum_{n=1}^{K_\Omega} \frac{\rho_{m,n}^+(\Omega) \hat{\bar{u}}_m^+(\Omega) \hat{u}_n^+(\Omega) + \rho_{m,n}^-(\Omega) \hat{\bar{u}}_m^-(\Omega) \hat{u}_n^-(\Omega)}{4 k_m k_n d_m d_n} \right) \pi \Omega d\Omega > 0, \tag{7.33}$$

where the last equality is due to Lemma 7.6. Similar we prove that the quadratic form $\langle \mathbf{u}^+, \mathcal{L}_0^{-1} \mathbf{u}^+ \rangle$ is positive definite for $\mathbf{u}^+ \in Y_c^{(u)}(\mathcal{A})$. ∎

8 Symmetry–breaking stability analysis

Stability analysis based on simultaneous diagonalization of two linear operators can be applied to other Hamiltonian dynamical systems, such as Klein–Gordon and Korteweg–de Vries equations. We show here that similar analysis is applied also to symmetry–breaking instabilities of stationary solutions of coupled NLS equations.

Symmetry-breaking instabilities may occur when the stationary solutions in (z, x) are perturbed in another spatial dimension, say in y [KP00]. The system of coupled NLS equations (1.1) is rewritten in three spatial dimensions (z, x, y) as:

$$i \frac{\partial \psi_n}{\partial z} + d_n \left(\frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} \right) + f_n(|\psi_1|^2, ..., |\psi_N|^2) \psi_n = 0, \qquad n = 1, .., N. \tag{8.1}$$

We assume the same conditions on f_n and d_n, as below (1.1). Linearization of the stationary solutions (2.1) is defined by the expansion,

$$\psi_n(z, x, y) = \left[\Phi_n(x) + U_n(z, x, y) + i W_n(z, x, y) \right] e^{i \beta_n z}, \tag{8.2}$$

where $(U_n, W_n)^T \in \mathbb{R}^2$ are perturbation functions. Separating the variables (z, x, y) as $\mathbf{U} = \mathbf{u}(x)\, e^{\lambda z + ipy}$, $\mathbf{W} = \mathbf{w}(x)\, e^{\lambda z + ipy}$, we arrive to the linear eigenvalue problem,

$$\left(\mathcal{L}_1 + p^2 \mathcal{D} \right) \mathbf{u} = -\lambda \mathbf{w}, \qquad \left(\mathcal{L}_0 + p^2 \mathcal{D} \right) \mathbf{w} = \lambda \mathbf{u}, \tag{8.3}$$

where $(\mathbf{u}, \mathbf{w})^T \in \mathbb{R}^{2N}$, $p \in \mathbb{R}$, and $\mathcal{D}$ is a diagonal matrix of $(d_1, ..., d_N)$. Eigenvalues λ and eigenvectors $(\mathbf{u}, \mathbf{w})^T$ of the linearized problem (8.3) depend on parameter p.

Lemma 8.1 *There exist exactly $n(\mathcal{L}_{1,0})$ negative eigenvalues λ of the problem*

$$\mathcal{L}_{1,0} \mathbf{u} = \lambda \mathcal{D} \mathbf{u}, \qquad \mathbf{u} \in L^2(\mathbb{R}), \qquad \lambda < 0. \tag{8.4}$$

Proof. Since $\mathcal{D}$ is positive definite, all eigenvalues λ in (8.4) are real. Proposition 6.1 suggests that the negative index of quadratic forms $\langle \mathbf{u}, \mathcal{L}_{1,0}\mathbf{u}\rangle$ in Hilbert space $L^2(\mathbb{R})$ is invariant in two diagonal representations, one with respect to $\langle \mathbf{u}_n, \mathbf{u}_n\rangle$ and the other one with respect to $\langle \mathbf{u}_n, \mathcal{D}\mathbf{u}_n\rangle > 0$. ∎

We define negative eigenvalues λ of the problem (8.4) for $\mathcal{L}_1$ as $\lambda = -\{\Lambda_{1,n}^2\}_{n=1}^{n(\mathcal{L}_1)}$ and for $\mathcal{L}_0$ as $\lambda = -\{\Lambda_{0,n}^2\}_{n=1}^{n(\mathcal{L}_0)}$. We split the domain $p^2 \in \mathbb{R}_+$ into sub-domains:

$$\mathcal{D}_{n_1,n_0} = \{p^2 \in \mathbb{R}_+ : \Lambda_{1,n_1}^2 < p^2 < \Lambda_{1,(n_1+1)}^2,\ \Lambda_{0,n_0}^2 < p^2 < \Lambda_{0,(n_0+1)}^2\}, \tag{8.5}$$

where $0 \le n_1 \le n(\mathcal{L}_1)$, $0 \le n_0 \le n(\mathcal{L}_0)$, such that $\Lambda_{1,0}^2 = \Lambda_{0,0}^2 \equiv 0$, and $\Lambda_{1,(n(\mathcal{L}_1)+1)}^2 = \Lambda_{0,(n(\mathcal{L}_0)+1)}^2 \equiv \infty$.

Lemma 8.2 *In the domain $\mathcal{D}_{n_1,n_0}$, there are exactly $(n(\mathcal{L}_1)-n_1)$ negative eigenvalues of the problem*

$$\left(\mathcal{L}_1 + p^2\mathcal{D}\right)\mathbf{u} = \lambda\mathbf{u}, \qquad \mathbf{u} \in L^2(\mathbb{R}), \qquad \lambda < 0, \tag{8.6}$$

and exactly $(n(\mathcal{L}_0)-n_0)$ negative eigenvalues of the problem,

$$\left(\mathcal{L}_0 + p^2\mathcal{D}\right)\mathbf{u} = \lambda\mathbf{u}, \qquad \mathbf{u} \in L^2(\mathbb{R}), \qquad \lambda < 0. \tag{8.7}$$

Proof. The result follows from continuity of eigenvalues λ of the uncoupled problems (8.6) and (8.7) with respect to parameter p^2 in the domain $0 \le p^2 < \infty$. Each negative eigenvalue $\lambda = \lambda(p^2)$ of operator $\left(\mathcal{L}_{1,0} + p^2\mathcal{D}\right)$ is an increasing function of p^2 if $\mathcal{D}$ is positive definite, since

$$\lambda'(p^2) = \frac{\langle \mathbf{u}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathcal{D}\mathbf{u}\rangle} > 0. \tag{8.8}$$

When p^2 increases, eigenvalues $\lambda(p^2)$ pass through the zero value at the boundaries between domains $\mathcal{D}_{n_1,n_0}$ in (8.5), and the number of negative eigenvalues of (8.6)–(8.7) reduces according to the multiplicity of eigenvalues $\lambda = -\Lambda_{1,n_1}^2$ and $\lambda = -\Lambda_{0,n_0}^2$ in (8.4). ∎

Proposition 8.3 *Assume that Assumption 2.15 is satisfied for the problem (8.3) in the domain $\mathcal{D}_{n_1,n_0}$. The linearized problem (8.3) has $N_{\rm unst} = N_{\rm real} + 2N_{\rm comp}$ unstable eigenvalues $\lambda = \lambda(p)$ with* $\mathrm{Re}(\lambda) > 0$, *such that*

$$\begin{aligned}
&(i) \quad |n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0| \le N_{\rm unst} \le (n(\mathcal{L}_1) + n(\mathcal{L}_0) - n_1 - n_0), && (8.9)\\
&(ii) \quad N_{\rm real} \ge |n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0|, && (8.10)\\
&(iii) \quad N_{\rm comp} \le \min\left(n(\mathcal{L}_0) - n_0, n(\mathcal{L}_1) - n_1\right). && (8.11)
\end{aligned}$$

Proof. The linearization problem (8.3) can be rewritten in the form of a diagonalization problem,

$$\left(\mathcal{L}_1 + p^2\mathcal{D}\right)\mathbf{u} = \gamma\left(\mathcal{L}_0 + p^2\mathcal{D}\right)^{-1}\mathbf{u}, \qquad \mathbf{u} \in L^2(\mathbb{R}), \tag{8.12}$$

where $\gamma = -\lambda^2$. If $p^2 > 0$ and $p^2 \ne \Lambda_{0,n}^2$, $n = 1, ..., n(\mathcal{L}_0)$, the operator $(\mathcal{L}_0 + p^2\mathcal{D})$ is invertible in $L^2(\mathbb{R})$. Proposition 6.1 applies with $\#_{<0}(\mathcal{L}_1 + p^2\mathcal{D}) = n(\mathcal{L}_1) - n_1$ and $\#_{<0}(\mathcal{L}_0 + p^2\mathcal{D}) = n(\mathcal{L}_0) - n_0$ in the domain D_{n_1,n_0}. Proposition 8.3 is then equivalent to Theorem 8. ∎

Proposition 8.4 *Assume that Assumption 2.15 is satisfied for the problem (2.14). Let $N_{\rm unst}$ be the number of unstable eigenvalues in the problem (2.14). There exists $p_*^2 > 0$ such that the linearized problem (8.3) has exactly $\hat{N}_{\rm unst}$ unstable eigenvalues in the domain $0 < p^2 < p_*^2$, where $\hat{N}_{\rm unst} = N_{\rm unst} + p(\mathcal{U})$. The new $p(\mathcal{U})$ unstable eigenvalues λ are all real and positive.*

Proof. Lemma 2.17 suggests that the linearized problem (2.14) has $N+1$ double zero eigenvalues in $L^2(\mathbb{R})$. The symmetry-breaking perturbation with $p^2 > 0$ split these double eigenvalues into pairs of real or imaginary eigenvalues $\lambda(p)$ of the linearized problem (8.3). We show that $p(\mathcal{U})$ double eigenvalues split into pairs of real eigenvalues λ, by perturbation series arguments [K76]. We expand solutions of (8.3) into power series of p as follows,

$$\begin{aligned} \mathbf{u} &= p\lambda_1 \sum_{n=1}^{N} c_n \frac{\partial \mathbf{\Phi}}{\partial \beta_n} + \mathrm{O}(p^3), \\ \mathbf{w} &= \sum_{n=1}^{N} c_n \Phi_n(x) \mathbf{e}_n + p^2 \mathbf{w}_2(x) + \mathrm{O}(p^4), \end{aligned} \tag{8.13}$$

where $\lambda = p\lambda_1 + \mathrm{O}(p^3)$. The function $\mathbf{w}_2(x)$ satisfies the non-homogeneous linear problem,

$$\mathcal{L}_0 \mathbf{w}_2 = \lambda_1^2 \sum_{n=1}^{N} c_n \frac{\partial \mathbf{\Phi}}{\partial \beta_n} - \sum_{n=1}^{N} c_n d_n \Phi_n(x) \mathbf{e}_n. \tag{8.14}$$

Using the Fredholm Alternative Theorem, we find that $\mathbf{c} = (c_1, ..., c_N)^T$ satisfies the generalized eigenvalue problem,

$$\lambda_1^2 \mathcal{U} \mathbf{c} = 2 \mathcal{D} \mathcal{Q}_s \mathbf{c}, \tag{8.15}$$

where $\mathcal{Q}_s$ is a diagonal matrix of $(Q_{1s}, ..., Q_{ns})^T$ and $\mathcal{U}$ is the Hessian matrix (2.7). Since $\mathcal{D}\mathcal{Q}_s$ is positive-definite, Sylvester's Inertia Theorem suggests that the linear system (8.15) has exactly $p(\mathcal{U})$ positive eigenvalues and $n(\mathcal{U})$ negative eigenvalues λ_1^2. Therefore, positive eigenvalues of $\mathcal{U}$ are related to new unstable (real and positive) eigenvalues $\lambda = \lambda(p)$ in the linearization problem (8.3) for sufficiently small values of $p^2 > 0$, in addition to $N_{\rm unst}$ unstable eigenvalues $\lambda(p)$ existing in the limit $p^2 \to 0$ with $\mathrm{Re}(\lambda) > 0$. ■

Proposition 8.4 agrees with Proposition 8.3 for $n_1 = 0$ and $n_0 = 0$. We also notice that the (N+1)-th double zero eigenvalue with the eigenvector $(\mathbf{\Phi}'(x), \mathbf{0}_N)^T$ splits into a pair of imaginary eigenvalues for $p^2 > 0$, due to Galileo invariance (1.7), since the translational symmetry (1.6) does not change the index $p(\mathcal{U}_H)$.

With a limited use, the same method is applied to the hyperbolic NLS equations, such as

$$i\psi_z + \psi_{xx} - \psi_{yy} + |\psi|^2 \psi = 0. \tag{8.16}$$

The stationary solutions are $\psi(z,x) = \sqrt{2\beta}\, \mathrm{sech}(\sqrt{\beta} x)\, e^{i\beta z}$, where $\beta > 0$. It is easy to check that $n(\mathcal{L}_1) = 1$, $n(\mathcal{L}_0) = 0$, and $z(\mathcal{L}_1) = z(\mathcal{L}_0) = 1$. In the domain $0 < p^2 < \beta$, zero eigenvalues of

$\mathcal{L}_1$ and $\mathcal{L}_0$ become negative eigenvalues in the problems (8.6) and (8.7). Proposition 8.3 applies with $n_1 = -1$ and $n_0 = -1$ and suggests that there are $1 \leq N_{\rm unst} \leq 3$ unstable eigenvalues in the linearized problem (8.3) for $0 < p^2 < \beta$. Indeed, it was shown in [P01] that there is one real positive eigenvalue $\lambda(p)$ in the domain $0 < p^2 < p_{\rm thr}^2$, where $p_{\rm thr}^2 < \beta$, and three unstable ($N_{\rm real} = 1$, $2N_{\rm comp} = 2$) eigenvalues $\lambda(p)$ for $p_{\rm thr}^2 < p^2 < \beta$. The stability analysis breaks in the domain $p^2 \geq \beta$, where the negative space of operators $(\mathcal{L}_1 + p^2\mathcal{D})$ and $(\mathcal{L}_0 + p^2\mathcal{D})$ becomes infinite-dimensional.

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